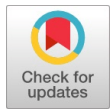




Every Nontrivial Finite MRB Partial Sum Is Irrational Lebsir Version

Marvin Ray Burns



Abstract: Let $S_N = \sum_{n=1}^N (-1)^n (n^{1/n} - 1)$ denote the N th partial sum of the defining series for the MRB constant. We prove that every nontrivial finite partial sum is an irrational algebraic number: $S_N \notin \mathbb{Q}$ ($N \geq 2$). The proof is entirely finite and algebraic. The proof uses the vanishing field trace of each radical $n^{1/n}$, together with the exact contribution of the rational terms in the alternating sum. We also interpret the fractional-part representation $S_N = \sum_{n=1}^N (-1)^n \text{Fractional Part} \left(n^{\frac{1}{n}} \right)$ within Habib Lebsir's congruence framework modulo $1/n$. The MRB sequence corresponds to the diagonal choice $x = n$ in Lebsir's family of partitions, for which $\lfloor n^{1/n} \rfloor = 1$ ($n \geq 2$). Thus, rationality never occurs at a finite stage of the defining MRB process. This does not determine the arithmetic nature of the infinite MRB constant. Still, it shows that any rationality of the limit, were it to occur, could arise only through the infinite limiting process and not through a finite algebraic cancellation among its defining radicals.

Keywords: MRB Constant, Algebraic Irrationality, Field Trace, Radical Extensions, Fractional Parts, Lebsir Congruences

I. INTRODUCTION

The conditionally convergent alternating series define the MRB constant.

$$C_{\text{MRB}} = \sum_{n=1}^{\infty} (-1)^n (n^{1/n} - 1) \\ = 0.1878596424620671202485179340542732300559 \dots$$

Its arithmetic nature remains unknown: no proof is presently known that C_{MRB} is rational, irrational, algebraic irrational, or transcendental.

An earlier investigation of the rationality and transcendental problem approached the constant from the analytic side [1]. That work introduced a family of Mellin-type integrals and an exact integration-by-parts ladder associated with the MRB series. It also clarified an important distinction between the discrete alternating sum defining C_{MRB} and a continuous contour analogue. A naive termwise passage from the exponential expansion

$$n^{1/n} - 1 = \sum_{k=1}^{\infty} \frac{(\log n)^k}{k! n^k}$$

The corresponding contour integrals do not reproduce the MRB constant itself. Abel regularisation is required to rigorously recover the original discrete sum.

The same analytic framework displayed explicit powers of π^{-1} throughout the integration-by-parts ladder and thereby suggested a possible structural obstruction to algebraicity. That part of the earlier argument was explicitly heuristic and conditional, however, and did not constitute a proof of either irrationality or transcendence of C_{MRB} .

A second perspective relevant to the present work comes from Habib Lebsir's theory of congruences modulo $1/n$ and positive integers modulo $1/n$ [2]. For a fixed positive integer n , Lebsir partitions the positive integers according to the integer part of their n th roots. Thus, the value

$$\lfloor x^{1/n} \rfloor$$

determines the $1/n$ -class containing x . This framework naturally interprets the elementary identity.

$$n^{1/n} - 1 = \text{FractionalPart}(n^{1/n}), \quad n \geq 2,$$

which is central to the fractional-part form of the MRB series.

The present paper takes a finite and substantially more elementary direction. Rather than asking immediately about the arithmetic nature of the infinite limit, we first ask:

Can rationality occur at any finite stage of the defining MRB series?

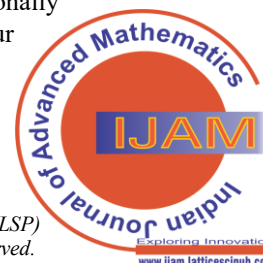
The answer is no.

For every $N \geq 2$, the partial sum S_N lies in the finite algebraic extension

$$L_N = \mathbb{Q}(2^{1/2}, 3^{1/3}, \dots, N^{1/N}),$$

so, its arithmetic can be studied directly by field-theoretic methods. We show that each radical $n^{1/n}$ has trace zero over $\mathbb{Q}$. Taking the trace of the entire partial sum then leaves only the rational alternating contribution. If S_N were rational, this forces it to equal either 0 or -1 , according to the parity of N . Elementary estimates and monotonicity exclude both possibilities.

Consequently, every nontrivial finite MRB partial sum is an irrational algebraic number. No limiting argument, rearrangement of a conditionally convergent series, contour deformation, or regularization is required.



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A. Lebsir's $1/n$ -Congruence Viewpoint

Lebsir begins with the elementary but useful fact that, for each fixed positive integer n and every positive integer x , there exists a unique nonnegative integer m such that

$$m^n \leq x < (m+1)^n.$$

Equivalently,

$$m = \lfloor x^{1/n} \rfloor.$$

This leads to an equivalence relation on the positive integers: two integers x and y are congruent in Lebsir's sense modulo $1/n$ when

$$\lfloor x^{1/n} \rfloor = \lfloor y^{1/n} \rfloor.$$

The corresponding equivalence class indexed by m may therefore be written

$$C_{m,n} = \{x \in \mathbb{N} : \lfloor x^{1/n} \rfloor = m\}.$$

Equivalently,

$$C_{m,n} = \{x \in \mathbb{N} : m^n \leq x < (m+1)^n\}.$$

Lebsir calls the elements of such a class its components, and the number of components its amplitude.

The defining MRB sequence selects a particularly simple diagonal through this two-parameter family of classes. Namely, take

$$x = n.$$

For every integer $n \geq 2$,

$$1^n < n < 2^n.$$

Therefore

$$\lfloor n^{1/n} \rfloor = 1.$$

Thus, in Lebsir's terminology, the integer n is a component of the class indexed by 1 modulo $1/n$. Indeed,

$$C_{1,n} = \{x \in \mathbb{N} : 1 \leq x^{1/n} < 2\} = \{1, 2, \dots, 2^n - 1\}.$$

Hence the amplitude of this class is

$$2^n - 1.$$

The particular component chosen by the MRB sequence is $x = n$. Subtracting the integer part of its n th root leaves exactly

$$n^{1/n} - \lfloor n^{1/n} \rfloor = n^{1/n} - 1.$$

Consequently,

$$n^{1/n} - 1 = \text{FractionalPart}(n^{1/n}) \quad (n \geq 2).$$

Thus, the fractional-part formulation of the MRB series can be viewed as the residual part obtained by following the diagonal.

$$x = n$$

through Lebsir's family of $1/n$ -classes:

$$n \mapsto C_{1,n} \mapsto n^{1/n} - \lfloor n^{1/n} \rfloor.$$

This observation does not supply the irrationality proof below; that proof is field-theoretic. It does, however, give a natural structural setting for the fractional-part representation and explains why the integer contribution to every nontrivial MRB term is exactly 1.

II. FINITE MRB PARTIAL SUMS

Define

$$S_N = \sum_{n=1}^N (-1)^n (n^{1/n} - 1).$$

Since the $n = 1$ term is zero,

$$S_1 = 0.$$

As the preceding section shows, for every integer $n \geq 2$,

$$\lfloor n^{1/n} \rfloor = 1,$$

and therefore

$$\text{FractionalPart}(n^{1/n}) = n^{1/n} - 1.$$

Thus

$$S_N = \sum_{n=1}^N (-1)^n \text{FractionalPart}(n^{1/n}).$$

The ordinary radical form and the fractional-part form therefore have the same finite partial sums.

Our main result is the following.

Theorem 1. *For every integer $N \geq 2$, the partial sum $S_N = \sum_{n=1}^N (-1)^n (n^{1/n} - 1)$ is an irrational algebraic number. In particular, $S_N \notin \mathbb{Q}$.*

The proof rests on a trace property of the individual radicals.

Lemma 1. *For every integer $n \geq 2$, $\text{Tr}_{\mathbb{Q}(n^{1/n})/\mathbb{Q}}(n^{1/n}) = 0$.*

Proof. Write the prime factorization of n as

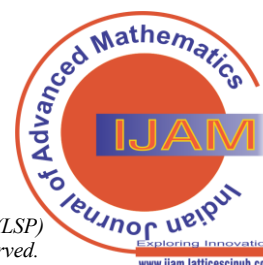
$$n = \prod_{j=1}^r q_j^{e_j},$$

and set

$$g = \gcd(n, e_1, \dots, e_r), \quad d = \frac{n}{g}, \quad c = \prod_{j=1}^r q_j^{e_j/g}.$$

Then

$$n^{1/n} = c^{1/d}.$$





Since $n \geq 2$, each exponent $e_j < n$, and hence $g < n$. Therefore $d > 1$.

By the definition of g , for every prime $\ell \mid d$, the integer c is not an ℓ th power in $\mathbb{Q}$. The standard irreducibility criterion for binomials therefore shows that the polynomial $x^d - c$ is irreducible over $\mathbb{Q}$.

When $4 \mid d$, the additional exceptional condition in that criterion is automatic here because $c > 0$.

Thus $x^d - c$ is the minimal polynomial of $n^{1/n}$. Since $d > 1$ and the coefficient of x^{d-1} is zero, the sum of its conjugates is zero. Therefore

$$\text{Tr}_{\mathbb{Q}(n^{1/n})/\mathbb{Q}}(n^{1/n}) = 0.$$

A. Proof of the Theorem

Fix $N \geq 2$, and let

$$L_N = \mathbb{Q}(2^{1/2}, 3^{1/3}, \dots, N^{1/N}),$$

with

$$D_N = [L_N : \mathbb{Q}].$$

For each $2 \leq n \leq N$, put

$$K_n = \mathbb{Q}(n^{1/n}).$$

Since $K_n \subseteq L_N$, trace transitivity gives

$$\text{Tr}_{L_N/\mathbb{Q}}(n^{1/n}) = [L_N : K_n] \text{Tr}_{K_n/\mathbb{Q}}(n^{1/n}) = 0.$$

Now

$$S_N = \sum_{n=2}^N (-1)^n n^{1/n} - \sum_{n=2}^N (-1)^n.$$

The first sum has trace zero. The elementary alternating sum satisfies.

$$\sum_{n=2}^N (-1)^n = \begin{cases} 0, & N \text{ odd,} \\ 1, & N \text{ even.} \end{cases}$$

Consequently,

$$\text{Tr}_{L_N/\mathbb{Q}}(S_N) = \begin{cases} 0, & N \text{ odd,} \\ -D_N, & N \text{ even.} \end{cases} \quad (1)$$

Suppose, for contradiction, that

$$S_N \in \mathbb{Q}.$$

For every rational number r ,

$$\text{Tr}_{L_N/\mathbb{Q}}(r) = D_N r.$$

Equation (1) would therefore force

$$S_N = \begin{cases} 0, & N \text{ odd,} \\ -1, & N \text{ even.} \end{cases} \quad (2)$$

It remains to exclude these two possibilities. Set

$$b_n = n^{1/n} - 1.$$

For $x > e$,

$$\frac{d}{dx} \left(\frac{\log x}{x} \right) = \frac{1 - \log x}{x^2} < 0.$$

Since

$$x^{1/x} = \exp \left(\frac{\log x}{x} \right),$$

the sequence b_n is strictly decreasing for integers $n \geq 3$.

B. Odd Partial Sums

First,

$$S_3 = \sqrt{2} - \sqrt[3]{3} < 0,$$

because

$$(\sqrt{2})^6 = 8 < 9 = (\sqrt[3]{3})^6.$$

Thus

$$S_3 \neq 0.$$

Next,

$$S_5 = 2\sqrt{2} - \sqrt[3]{3} - \sqrt[5]{5}.$$

We use the rational estimates.

$$\sqrt{2} > \frac{707}{500}, \quad \sqrt[3]{3} < \frac{1443}{1000}, \quad \sqrt[5]{5} < \frac{69}{50}.$$

These follow respectively from

$$707^2 < 2(500)^2, \quad 1443^3 > 3(1000)^3, \quad 69^5 > 5(50)^5.$$

Hence

$$S_5 > 2 \left(\frac{707}{500} \right) - \frac{1443}{1000} - \frac{69}{50} = \frac{1}{200} > 0.$$

For every odd $N \geq 5$,

$$S_N = S_5 + (b_6 - b_7) + (b_8 - b_9) + \dots + (b_{N-1} - b_N).$$

Every difference in parentheses is positive because b_n is strictly decreasing. Therefore

$$S_N > 0 \quad (N \geq 5, N \text{ odd}).$$

Thus, no odd partial sum S_N with $N \geq 3$ is zero.

C. Even Partial Sums

For $N = 2$,

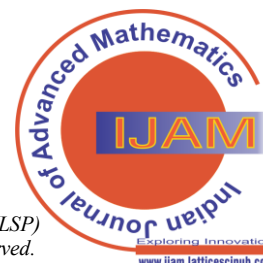
$$S_2 = \sqrt{2} - 1 > 0.$$

Also,

$$S_4 = S_5 + b_5 > S_5 > 0.$$

For every even $N \geq 6$,

$$S_N = S_{N-1} + b_N.$$



Since $N - 1 \geq 5$ is odd, the preceding argument gives $S_{N-1} > 0$, while $b_N > 0$. Hence

$$S_N > 0.$$

Therefore, every even partial sum is positive, and in particular.

$$S_N \neq -1.$$

Both rational alternatives in (2) are impossible. Thus

$$S_N \notin \mathbb{Q} \quad (N \geq 2).$$

Finally, $S_N \in L_N$, and $L_N/\mathbb{Q}$ is a finite algebraic extension. Hence S_N is algebraic. It follows that every nontrivial finite partial sum is an irrational algebraic number.

III. FRACTIONAL-PART FORMULATION

Corollary 1. *Every nontrivial finite partial sum $\sum_{n=1}^N (-1)^n \text{FractionalPart}(n^{1/n})$ is irrational for $N \geq 2$.*

Proof. For every $n \geq 2$, the diagonal $x = n$ lies in Lebsir's class $C_{1,n}$, and therefore

$$\lfloor n^{1/n} \rfloor = 1.$$

Hence

$$\text{FractionalPart}(n^{1/n}) = n^{1/n} - 1.$$

The fractional-part partial sum is exactly S_N , so the theorem applies directly.

A. What the Finite Theorem Says About the Infinite Problem

The theorem establishes an exact arithmetic property of every finite MRB approximation:

$$S_N \notin \mathbb{Q} \quad (N \geq 2).$$

Indeed, more strongly, S_N is algebraic and irrational ($N \geq 2$).

Thus, a sequence of irrational algebraic numbers approaches the MRB constant, and no nontrivial finite truncation of its defining series is rational.

This result does not prove that

$$C_{\text{MRB}} = \lim_{N \rightarrow \infty} S_N$$

is irrational. A convergent sequence of irrational algebraic numbers can have a rational limit. The finite theorem therefore cannot, by itself, decide the arithmetic nature of C_{MRB} .

It does, however, identify precisely where any hypothetical rationality would have to occur. If

$$C_{\text{MRB}} \in \mathbb{Q},$$

then rationality cannot arise from any finite cancellation among the radicals

$$2^{1/2}, 3^{1/3}, \dots, N^{1/N}.$$

It could occur only through the infinite limiting process.

"This gives a useful separation between two arithmetic questions. The finite problem is settled: S_N is an algebraic irrational number for every integer $N \geq 2$. The infinite

problem remains open: whether C_{MRB} is rational has not been determined.

The distinction also complements the earlier analytic investigation [1]. There the difficulty enters through conditional convergence, regularization, and genuinely infinite relations among analytic quantities. Here none of those issues is present: at every finite stage the arithmetic obstruction is exact.

Accordingly, any eventual proof concerning the irrationality or transcendence of C_{MRB} must use information that survives, or arises from, passage to the infinite limit. The finite algebra alone already excludes rationality at every finite stage.

IV. CONCLUSION

We have proved that every nontrivial finite partial sum of the defining MRB series is an irrational algebraic number. More precisely, if

$$S_N = \sum_{n=1}^N (-1)^n (n^{1/n} - 1),$$

then

$$S_N \notin \mathbb{Q} \quad (N \geq 2).$$

The proof is entirely finite and algebraic. Each radical $n^{1/n}$ has trace zero over $\mathbb{Q}$, so taking the field trace of S_N eliminates all of the radical contributions and leaves only the elementary alternating rational part. If S_N were rational, the trace calculation would force $S_N = 0$ for odd N or $S_N = -1$ for even N . Direct estimates and the monotonicity of $n^{1/n} - 1$ exclude both possibilities. Since S_N belongs to a finite algebraic extension of $\mathbb{Q}$, it follows that S_N is not merely irrational but algebraic irrational.

The fractional-part formulation gives the same conclusion. For every $n \geq 2$,

$$\lfloor n^{1/n} \rfloor = 1,$$

and hence

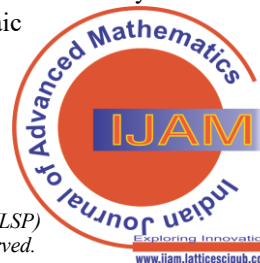
$$\text{FractionalPart}(n^{1/n}) = n^{1/n} - 1.$$

In Lebsir's $1/n$ -congruence framework, this corresponds to following the diagonal choice $x = n$ through the class $C_{1,n}$. Thus, Lebsir's partition provides a natural structural interpretation of the integer part removed from every MRB term, while the field-trace argument supplies the irrationality theorem itself.

The principal implication is a sharp separation between the finite and infinite rationality problems. Rationality is impossible at every nontrivial finite stage of the MRB construction. Therefore, if the MRB constant

$$C_{\text{MRB}} = \lim_{N \rightarrow \infty} S_N$$

were nevertheless rational, that rationality could not result from any finite algebraic cancellation among the radicals appearing in the defining series. It would





have to arise only through the genuinely infinite limiting process.

Accordingly, the finite arithmetic question is settled, while the arithmetic nature of C_{MRB} itself remains open. The present result narrows that problem by identifying where any future proof must look: not at a finite truncation, but at structure that persists or emerges only as $N \rightarrow \infty$. In this sense, the theorem provides both a complete answer to the finite problem and a precise boundary for the remaining rationality, irrationality, and transcendence questions surrounding the MRB constant.

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DECLARATION STATEMENT

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AUTHOR'S PROFILE



Marvin Ray Burns is a mathematics researcher whose work has centred on the MRB constant and the computational, analytic, and arithmetic questions arising from its defining alternating series. The MRB constant, introduced by Burns, is $C_{MRB} = \sum_{n=1}^{\infty} (-1)^n (n^{1/n} - 1)$. His investigations have included high-precision computation of the constant, convergence acceleration, analytic and contour representations, rationality and transcendentality questions, radical structure, and the arithmetic behaviour of finite MRB partial sums. The present work establishes that every nontrivial finite MRB partial sum is an irrational algebraic number and examines the fractional-part representation through Habib Lebsir's congruence framework. Burns continues to investigate the distinction between the now-settled finite rationality problem and the still-open arithmetic nature of the infinite MRB constant.

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