



Computational Mathematics and Structural Properties of Images of Fs-Subsets under an Fs-set Function

Ch. Ramasanyasi Rao, K. V. Uma Kameswari



Abstract: This paper investigates the computational mathematics of Fs-subsets within the refined framework of Complete Boolean algebras. Using computable homomorphisms, we establish key mathematical properties such as the Fs-De Morgan laws and structural projections. Finally, we discuss practical applications of those computational paradigms in multi-criteria decision-making fuzzy soft topology.

Keywords: Fs-set, Fs-subset, Complete Boolean Algebra, Fs-set function, Image and Inverse Image.

I. INTRODUCTION

Traditional fuzzy set theory initiated by L.A. Zadeh is restricted to membership values in the unit interval $[0,1]$. To overcome certain algebraic limitations, such as the validity of the Axiom of Choice for fuzzy sets, researchers Vaddipathi Yogeswara, etc.[1] introduced Fs-sets, where membership values reside in a complete Boolean Algebra(L). This paper explores Fs-set functions and computes the image and Inverse Image of Fs-subsets [2]. The theory of Fs-sets has since been developed by investigating Fs-subsets, Fs-set functions, and their associated images and inverse images. In particular, the concepts of image and inverse image of Fs-subsets under an Fs-set function provide important tools for studying the transformation of Fs-information between different underlying spaces [2], [3]. These investigations establish fundamental relationships involving inclusion, equality, and other set-theoretic properties of Fs-subsets and their images. Further developments have considered different types of Fs-set functions and examined their corresponding images and inverse images [4]. The study of Fs-sets is also related to broader developments in fuzzy and soft set theory. Since the introduction of fuzzy sets, researchers have proposed several generalisations to represent uncertainty, vagueness, and parameterised information.

In particular, soft sets and fuzzy soft sets have provided useful mathematical frameworks for dealing with uncertainty involving multiple parameters [5]. Researchers have studied the theory from various perspectives, including set-theoretic operations, functions, topology, compactness, and structural properties.

A. Fs-set

Definition: Let U be a universal set and $G \subseteq U$ be a non-empty set. A four-tuple $\mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), A_G)$ is said to be an Fs-set if and only if

- (1) $G \subseteq G_1 \subseteq U$
- (2) L_G is a complete Boolean Algebra
- (3) $\mu_{1G_1}: G_1 \longrightarrow A_G, \mu_{2G}: G \longrightarrow A_G$ are functions such that $\mu_{1A_1}|G \geq \mu_{2G}$
- (4) $\bar{G}: G \longrightarrow A_G$ is defined by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c$ for each $x \in G$

B. Fs-Subset

Definition: Let $\mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), A_G)$ and $\mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), A_H)$ be a pair of Fs-sets. $\mathcal{H}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{H} \subseteq \mathcal{G}$, if and only if

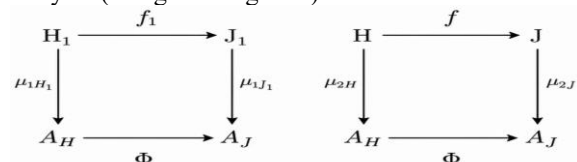
- (1) $H_1 \subseteq G_1, G \subseteq H$
- (2) A_H is a complete subalgebra of A_G or $A_H \leq A_G$
- (3) $\mu_{1H_1} \leq \mu_{1G_1}|H_1$, and $\mu_{2H}|G \geq \mu_{2G}$

Remark: For some A_X , such that $A_X \leq A_G$ a four-tuple $\mathcal{X} = (X_1, X, \bar{X}(\mu_{1X_1}, \mu_{2X}), A_X)$ is not an Fs-set if and only if

- (a) $X \not\subseteq X_1$ (or) X is an empty set
- (b) $\mu_{1X_1}x \not\geq \mu_{2X}x$, for some $x \in X \cap X_1$ or μ_{2X} is an empty function

C. Fs-Set Function

Definition: A Triplet (f_1, f, Φ) is said to be an Fs-set function between two given Fs-subsets $\mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), A_H)$ and $\mathcal{J} = (J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), A_J)$ of $\mathcal{H}$, denoted by $(f_1, f, \Phi): \mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), A_H) \rightarrow \mathcal{J} = (J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), A_J)$ if and only if (using the diagrams).



(Fig.1: Fs-set Function $\bar{f}: \mathcal{H} \rightarrow \mathcal{J}$)

Manuscript received on 21 August 2026 | First Revised Manuscript received on 12 September 2026 | Second Revised Manuscript received on 22 September 2026 | Manuscript Accepted on 15 October 2026 | Manuscript published on 30 October 2026.

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- (a) $f = f_1|_H^J: H \longrightarrow J$ be onto
 (b) $\Phi: A_H \longrightarrow A_J$ is a complete homomorphism
 (f_1, f, Φ) is denoted by $\bar{f}$

D. Fs- Image

Definition Let $\mathcal{K} \subseteq \mathcal{H}$ and $\bar{f}: \mathcal{H} \longrightarrow \mathcal{J}$ be an Fs-set function, where

$$\mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H), \mathcal{J} = (J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), L_J)$$

$$\mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K),$$

$K = H$ and $f = f_1|_H^J: H \longrightarrow J$ be onto.

Define the image of an Fs-subset $\mathcal{K}$ under the Fs-set function $\bar{f}$, denoted by $\bar{f}(\mathcal{K})$ as follows.

$\bar{f}(\mathcal{K}) = \mathcal{L} = (L_1, L, \bar{L}(\mu_{1L_1}, \mu_{2L}), A_L)$, where

- (1) $L_1 = f_1(K_1)$
- (2) $L = f(K) = f(H) = J$
- (3) $\mu_{1L_1}: L_1 \longrightarrow A_J$ is defined by

$$\mu_{1L_1}y = \begin{cases} \mu_{2J}yV \left[\mu_{1J_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in D_1}} \Phi \mu_{1K_1}x \right) \right], & \text{if } y \in J \\ \mu_{1J_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in K_1}} \Phi \mu_{1K_1}x \right), & \text{if } y \notin J \end{cases}$$

- (4) $\mu_{2L}: L \longrightarrow A_J$ is defined by

$$\mu_{2L}y = \mu_{2J}yV \left[\mu_{1J_1}y \wedge \left(\bigvee_{\substack{y=fx \\ x \in K}} \Phi \mu_{2K}x \right) \right]$$

- (5) $A_L = ([\mu_{1L_1}(L_1)])$ = The complete subalgebra generated by $[\mu_{1L_1}(L_1)]$, where $[\mu_{1L_1}(L_1)]$ = The complete ideal generated by $\mu_{1L_1}(L_1)$ in A_J

E. Fs-Inverse Image

Definition:

Let $\mathcal{K} \subseteq \mathcal{H}$ and $\bar{f}: \mathcal{H} \longrightarrow \mathcal{J}$ be an Fs- set function, $f = f_1|_H^J: H \longrightarrow J$ be onto.

-Let $\mathcal{L} \subseteq \mathcal{J}$, where $\mathcal{L} = (L_1, L, \bar{L}(\mu_{1L_1}, \mu_{2L}), A_L)$ such that $L=J$. Define an inverse image of the Fs-subset $\mathcal{L}$ under the Fs-set function $\bar{f}$, denoted by $\bar{f}^{-1}(\mathcal{L})$ as follows.

$\bar{f}^{-1}(\mathcal{L}) = \mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

- a) $K_1 = f_1^{-1}(L_1)$
- b) $K = f^{-1}(L) = f^{-1}(J) = H$
- c) $\mu_{1K_1}: K_1 \longrightarrow A_K$ is defined by

$$\mu_{1K_1}x = \begin{cases} \mu_{1H_1}x, & \text{whenever } \Phi^{-1}\mu_{1L_1}f_1x = \Phi \\ \mu_{2H}xV \left[\mu_{1H_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1L_1}f_1x \\ \alpha \in A_H}} \Phi^{-1}\mu_{1L_1}f_1x \right) \right], & x \in H \\ \mu_{1H_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1L_1}f_1x \\ \alpha \in A_H}} \Phi^{-1}\mu_{1L_1}f_1x \right), & x \notin H \end{cases}$$

- d) $\mu_{2K}: K \longrightarrow A_K$ is defined by

$$\mu_{2K}x = \begin{cases} \mu_{2H}x, & \text{whenever } \Phi^{-1}\mu_{2L}fx = \Phi \\ \mu_{2H}xV \left[\mu_{1H_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2L}fx \\ \alpha \in L_H}} \Phi^{-1}\mu_{2L}fx \right) \right] \end{cases}$$



e) $L_K = L_H$

II. MATHEMATICAL PRELIMINARIES

Before discussing the Fs-set function, we formally state the base structures.

Complete Boolean algebra:

Fs-set function:

Computational Mathematics of Images of Fs-Subsets:

Main Results: Resultant Properties of Images:

Theorem 1: If $\bar{f}: \mathcal{U} \rightarrow \mathcal{V}$ is an Fs-Set function and $\mathcal{P}_1, \mathcal{P}_2$ are Fs-subsets of $\mathcal{U}$ then

$$\bar{f}(\mathcal{P}_1 \cup \mathcal{P}_2) = \bar{f}(\mathcal{P}_1) \cup \bar{f}(\mathcal{P}_2)$$

Proof: Let $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

- (1) $K_1 = H_{11} \cup H_{12}$
- (2) $K = H_1 \cap H_2 = H$
- (3) $L_K = L_{H_1} \vee L_{H_2}$
- (4) $\mu_{1K_1}: K_1 \rightarrow L_K$ is defined by

$$\mu_{1K_1}x = (\mu_{1H_{11}} \vee \mu_{1H_{12}})x$$

- (5) $\mu_{2K}: K \rightarrow L_K$ is defined by

$$\mu_{2K}x = (\mu_{2H_1} \wedge \mu_{2H_2})x$$

and

$$\bar{f}(\mathcal{H}_1 \cup \mathcal{H}_2) = \bar{f}(\mathcal{K}) = \mathcal{M} = (M_1, M, \bar{M}(\mu_{1M_1}, \mu_{2M}), L_M)$$

- (6) $M_1 = f_1(K_1) = f_1(H_{11} \cup H_{12}) = f_1(H_{11}) \cup f_1(H_{12})$
- (7) $M = f(K) = f(H_1 \cap H_2) = f(B)$
- (8) $L_M = ([\mu_{1M_1}(M_1)])$
- (9) $\mu_{1M_1}: M_1 \rightarrow L_M$ is defined by

$$\mu_{1M_1}y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in K_1}} \Phi \mu_{1K_1}x \right) \right], & \text{if } y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in K_1}} \Phi \mu_{1K_1}x \right), & \text{if } y \notin C \end{cases}$$

- (10) $\mu_{2M}: M \rightarrow L_M$ is defined by

$$\mu_{2M}y = \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=fx \\ x \in K}} \Phi \mu_{2K}x \right) \right]$$

Suppose, $\bar{f}(\mathcal{H}_1) = \mathcal{G}_1 = (G_{11}, G_1, \bar{G}_1(\mu_{1G_{11}}, \mu_{2G_1}), L_{G_1})$, where

- (11) $G_{11} = f_1(H_{11})$
- (12) $G_1 = f(H_1) = f(B)$
- (13) $L_{G_1} = ([\mu_{1G_{11}}(G_{11})])$
- (14) $\mu_{1G_{11}}: G_{11} \rightarrow L_{G_1}$ is defined by

$$\mu_{1G_{11}}y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11}}} \Phi \mu_{1H_{11}}x \right) \right], & \text{if } y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11}}} \Phi \mu_{1H_{11}}x \right), & \text{if } y \notin C \end{cases}$$

- (15) $\mu_{2G_1}: G_1 \rightarrow L_{G_1}$ is defined by

$$\mu_{2G_1}y = \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_1}} \Phi \mu_{1H_1}x \right) \right]$$

and

$\bar{f}(\mathcal{H}_2) = \mathcal{G}_2 = (G_{12}, G_2, \bar{G}_2(\mu_{1G_{12}}, \mu_{2G_2})L_{G_2})$, where

$$(16) \quad G_{12} = f_1(H_{12})$$

$$(17) \quad G_2 = f(H_2) = f(B)$$

$$(18) \quad L_{G_2} = ([\mu_{1G_{12}}(G_{12})])$$

$$(19) \quad \mu_{1G_{12}}: G_{12} \rightarrow L_{G_2} \text{ is defined by}$$

$$\mu_{1G_{12}}y = \begin{cases} \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{12}}} \Phi \mu_{1H_{12}}x \right) \right], & \text{if } y \in C \\ \mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{12}}} \Phi \mu_{1H_{12}}x \right), & \text{if } y \notin C \end{cases}$$

$$(20) \quad \mu_{2G_2}: G_2 \rightarrow L_{G_2} \text{ is defined by}$$

$$\mu_{2G_2}y = \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_2}} \Phi \mu_{2H_2}x \right) \right]$$

Let $\bar{f}(\mathcal{H}_1) \cup \bar{f}(\mathcal{H}_2) = \mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{N} = (N_1, N, \bar{N}(\mu_{1N_1}, \mu_{2N}), L_N)$, where

$$(21) \quad N_1 = G_{11} \cup G_{12} = f_1(H_{11}) \cup f_1(H_{12}) = f_1(H_{11} \cup H_{12})$$

$$(22) \quad N = G_1 \cap G_2 = f(H_1) \cap f(H_2) = f(B) = f(H_1 \cap H_2)$$

$$(23) \quad L_N = L_{G_1} \vee L_{G_2} = ([\mu_{1G_{11}}(G_{11})]) \vee ([\mu_{1G_{12}}(G_{12})])$$

$$(24) \quad \mu_{1N_1}: N_1 \rightarrow L_N \text{ is defined as } \mu_{1N_1}y = (\mu_{1G_{11}} \vee \mu_{1G_{12}})y$$

$$(25) \quad \mu_{2N}: N \rightarrow L_N \text{ is defined as } \mu_{2N}y = \mu_{2G_1}y \wedge \mu_{2G_2}y$$

Need to show $\mathcal{M} = \bar{f}(\mathcal{H}_1 \cup \mathcal{H}_2)$ and $\mathcal{N} = \bar{f}(\mathcal{H}_1) \cup \bar{f}(\mathcal{H}_2)$ are 3-equal i.e. to show

$$(26) \quad M_1 = N_1, M = N$$

$$(27) \quad \mu_{1M_1} = \mu_{1N_1}$$

$$(28) \quad L_M = L_N$$

Proof of (26) follows from (21), (22), (6) and (7)

Proof of (27) for $y \in C$,

$$\begin{aligned} \mu_{1M_1}y &= \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in K_1}} \Phi \mu_{1K_1}x \right) \right] \\ &= \mu_{2C}y \vee \left[\mu_{1C_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11} \cup H_{12}}} \Phi(\mu_{1H_{11}} \vee \mu_{1H_{12}})x \right) \right] \end{aligned}$$

Consider the particular term in the above expression

$$\begin{aligned} &\bigvee_{\substack{y=f_1x \\ x \in H_{11} \cup H_{12}}} \Phi(\mu_{1H_{11}} \vee \mu_{1H_{12}})x \\ &= \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}}x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_2}} \Phi \mu_{1H_{12}}x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_3}} (\Phi \mu_{1H_{11}}x \vee \Phi \mu_{1H_{12}}x) \right), \text{ when } H_{11} \cup H_{12} = X_1 \cup X_2 \cup X_3, \end{aligned}$$

where

$$X_1 = H_{11} - H_{12}, X_2 = H_{12} - H_{11}, X_3 = H_{11} \cap H_{12} \text{ and}$$

For $y \in C$, we have $y \in G_{11}$ and $y \in G_{12}$

$$\mu_{1N_1}y = (\mu_{1G_{11}} \vee \mu_{1G_{12}})y$$

$$= \mu_{1G_{11}}y \vee \mu_{1G_{12}}y$$



$$\begin{aligned}
 &= \left[\mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11}}} \Phi \mu_{1H_{11}} x \right) \right] \right] \vee \left[\mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{12}}} \Phi \mu_{1H_{12}} x \right) \right] \right] \\
 &= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11}}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in H_{12}}} \Phi \mu_{1H_{12}} x \right) \right\} \right] \\
 &= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_3}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_3}} \Phi \mu_{1H_{12}} x \right) \right\} \right] \\
 &= \mu_{2C} y \vee \left[\mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_3}} (\Phi \mu_{1H_{12}} x \vee \Phi \mu_{1H_{11}} x) \right) \right\} \right]
 \end{aligned}$$

so that

$$\mu_{1M_1} = \mu_{1N_1} \text{ on } C$$

For $y \notin C$ and $y \in M_1 = f_1(H_{11} \cup H_{12}) = f_1(H_{11}) \cup f_1(H_{12})$

$$\begin{aligned}
 \mu_{1M_1} y &= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11} \cup H_{12}}} \Phi(\mu_{1H_{11}} \vee \mu_{1H_{12}}) x \right) \\
 &= \mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1x \\ x \in X_3}} (\Phi \mu_{1H_{11}} x \vee \Phi \mu_{1H_{12}} x) \right) \right\}
 \end{aligned}$$

For $y \notin C$, $y \in G_{11}$ and $y \notin G_{12} = f_1(H_{12})$ we have

$$\mu_{1M_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11} \cup H_{12}}} \Phi(\mu_{1H_{11}} \vee \mu_{1H_{12}}) x \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right)$$

$$\text{and } \mu_{1N_1} y = (\mu_{1G_{11}} \vee \mu_{1G_{12}}) y,$$

$$= \mu_{1G_{11}} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11}}} \Phi \mu_{1H_{11}} x \right)$$

$$= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right), \text{ so that}$$

$$\mu_{1M_1} = \mu_{1N_1}$$

For $y \notin C$, $y \in G_{12}$ and $y \notin G_{11} = f_1(H_{11})$ we have

$$\mu_{1M_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1x \\ x \in H_{11} \cup H_{12}}} \Phi(\mu_{1H_{11}} \vee \mu_{1H_{12}}) x \right)$$

$$\begin{aligned}
 &= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \\
 &\text{and } \mu_{1N_1} y = (\mu_{1G_{11}} \vee \mu_{1G_{12}}) y \\
 &= \mu_{1G_{12}} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{12}}} \Phi \mu_{1H_{12}} x \right) \\
 &= \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \text{ so that} \\
 &\quad \mu_{1M_1} = \mu_{1N_1} \\
 &\text{For } y \notin C, y \in G_{12} \text{ and } y \in G_{12} \text{ we have} \\
 &\mu_{1M_1} y = \mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{11} \cup H_{12}}} \Phi (\mu_{1H_{11}} \vee \mu_{1H_{12}}) x \right) \\
 &= \mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1 x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_3}} (\Phi \mu_{1H_{11}} x \vee \Phi \mu_{1H_{12}} x) \right) \right\} \\
 &\quad \text{and } \mu_{1N_1} y = (\mu_{1G_{11}} \vee \mu_{1G_{12}}) y \\
 &\quad = \mu_{1G_{11}} y \vee \mu_{1G_{12}} y \\
 &= \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{11}}} \Phi \mu_{1H_{11}} x \right) \right] \vee \left[\mu_{1C_1} y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{12}}} \Phi \mu_{1H_{12}} x \right) \right] \\
 &= \mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{11}}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in H_{12}}} \Phi \mu_{1H_{12}} x \right) \right\} \\
 &= \mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1 x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_3}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_3}} \Phi \mu_{1H_{12}} x \right) \right\} \\
 &= \mu_{1C_1} y \wedge \left\{ \left(\bigvee_{\substack{y=f_1 x \\ x \in X_1}} \Phi \mu_{1H_{11}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_2}} \Phi \mu_{1H_{12}} x \right) \vee \left(\bigvee_{\substack{y=f_1 x \\ x \in X_3}} (\Phi \mu_{1H_{12}} x \vee \Phi \mu_{1H_{11}} x) \right) \right\} \text{ so that} \\
 &\quad \mu_{1M_1} = \mu_{1N_1}
 \end{aligned}$$

Hence, in any case, $y \in G_{11}$, $y \notin G_{12}$ or $y \in G_{12}$, $y \notin G_{11}$ or $y \in G_{11}$, $y \in G_{12}$ we can have

$\mu_{1M_1} = \mu_{1N_1}$ for each $y \in M_1 = N_1$

Proof of (28)

In the following proof, we can see that

$$L_M = ([(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})])$$

We have $L_M = ([\mu_{1M_1}(M_1)])$

$$\begin{aligned}
 &= ([\mu_{1M_1}(f_1(H_{11} \cup H_{12}))]) \\
 &= ([\mu_{1M_1}(f_1(H_{11}) \cup f_1(H_{12}))]) \\
 &= ([\mu_{1M_1}(f_1(H_{11})) \cup \mu_{1M_1}(f_1(H_{12}))]) \\
 &= ([\mu_{1N_1}(f_1(H_{11})) \cup \mu_{1N_1}(f_1(H_{12}))]) \\
 &= ([(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})])
 \end{aligned}$$

Now we show that $L_M = L_N$





But $L_N = L_{G_1} \vee L_{G_2} = ([\mu_{1G_{11}}(G_{11})]) \vee ([\mu_{1G_{12}}(G_{12})])$
 For $y \in G_{11}$,
 $\mu_{1G_{11}}y \leq (\mu_{1G_{11}} \vee \mu_{1G_{12}})y \in [(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})]$
 Implying $\mu_{1G_{11}}(G_{11}) \subseteq [(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})]$
 again implying $[\mu_{1G_{11}}(G_{11})] \subseteq [(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})]$
 imply $([\mu_{1G_{11}}(G_{11})]) \leq ([(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})])$
 Similarly, for $y \in G_{12}$
 $([\mu_{1G_{12}}(G_{12})]) \leq ([(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})])$
 So that,
 $([\mu_{1G_{11}}(G_{11})]) \vee ([\mu_{1G_{12}}(G_{12})]) \leq ([(\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})])$
 $L_N \leq L_M$
 Sufficient to prove $L_M \leq L_N$
 Case1: $y \in G_{11}$ and $y \notin G_{12}$
 $(\mu_{1G_{11}} \vee \mu_{1G_{12}})y = \mu_{1G_{11}}y \in (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \subseteq (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})$, so that $\mu_{1G_{11}}y \in \mu_{1G_{11}}G_{11} \subseteq [(\mu_{1G_{11}})(G_{11})] \subseteq ([(\mu_{1G_{11}})(G_{11})])$
 Case2: $y \in G_{12}$ and $y \notin G_{11}$
 $(\mu_{1G_{11}} \vee \mu_{1G_{12}})y = \mu_{1G_{12}}y \in (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12}) \subseteq (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})$ so that, $\mu_{1G_{12}}y \in (\mu_{1G_{12}})(G_{12}) \subseteq [(\mu_{1G_{11}})(G_{12})] \subseteq ([(\mu_{1G_{11}})(G_{12})])$
 Case3: $y \in G_{12}$ and $y \in G_{11}$
 $(\mu_{1G_{11}} \vee \mu_{1G_{12}})y = \mu_{1G_{12}}y \vee \mu_{1G_{11}}y \in (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})$ so that
 $\mu_{1G_{12}}y \vee \mu_{1G_{11}}y \in [\mu_{1G_{11}}(G_{11})] \cup [(\mu_{1G_{11}})(G_{12})] \subseteq ([\mu_{1G_{11}}(G_{11})]) \vee ([\mu_{1G_{12}}(G_{12})])$
 From all Case 1, Case 2 and Case 3, we can conclude that.
 $([\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{11}) \cup (\mu_{1G_{11}} \vee \mu_{1G_{12}})(G_{12})]) \subseteq ([\mu_{1G_{11}}(G_{11})]) \vee ([\mu_{1G_{12}}(G_{12})])$
 so that

$$L_M \leq L_N$$

III. ISOMORPHISM

Any algebra homomorphism f is said to be an isomorphism if f is one-to-one and onto

A. Fs-Fixed Point

Definition: Let $\bar{f} = (f_1, f, \Phi): \mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H) \rightarrow \mathcal{J} = (J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), L_J)$. An Fs- Subset $\mathcal{C} \subseteq \mathcal{B}$ is called an Fs-fixed point of the function if $\bar{f}(x^*) = x^*$.

Remark: The Fs-subset remains invariant under the transformation $\bar{f}$.

Eg: Let $A = \{a_1, a_2, a_3, a_4\}$

Define Fs-set $\mathcal{X} = \{(a_1, (1,0)), (a_2, (1,1)), (a_3, (0,1)), a_4, (1,1)\}$. Suppose the Fs-set function is $\bar{f}(X) = X$. Then X itself is an Fs-fixed point because $\bar{f}(X) = X$.

B. Types of Fs-Fixed Points

- Stable Fs- fixed point:**
Small changes in Fs-membership converge back to the fixed point.
- Unstable Fs-fixed point:**
Small perturbations move away from the fixed point
- Global Fs- fixed point:**
For every initial Fs-subset $\mathcal{X}_k \rightarrow \mathcal{X}^*$

C. Metric Form of Fs-Fixed Point:)

If d is an Fs-distance metric, then $d(\bar{f}(X^*), X^*) = 0$

Computational Interpretation:

- Stable knowledge states
- Converged learning structures
- Stable uncertainty models

- Recurrent AI states.

Entropy-stable Fs-fixed point:

Definition: An Fs-fixed point $\mathcal{X}^*$ is called entropy stable if $\mathcal{R}(\bar{f}(\mathcal{X}^*)) = \mathcal{R}(\mathcal{X}^*)$.

This connects fixed-point theory and Computational AI Systems.

IV. RESULTS

A. Fs-Fixed Point Theorem

Suppose $\bar{f} = (f_1, f, \Phi): \mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H) \rightarrow \mathcal{H} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$ Then $\bar{f}(X^*) = X^*$.

Proof: Since $\mathcal{H}$ forms a finite complete lattice and $\bar{f}$ is monotone.

By the Knaster-Tarski principle, the set $\text{Fix}(\bar{f})$ is such that $\bar{f}(X^*) = X^*$.

Define an iterative sequence;

$$X_2 = \bar{f}(X_1)$$

$$X_3 = \bar{f}(X_2)$$

⋮

$$X_{n+1} = \bar{f}(X_n)$$

Because the lattice is finite

$$X_0 \leq X_1 \leq X_2 \leq \dots \leq \text{ie } X_k = \bar{f}(X_k)$$

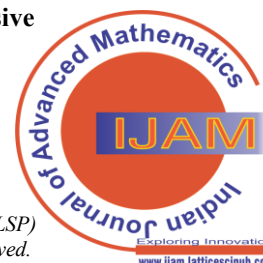
B. Fs-Convergence Theorem

Let $X_{n+1} = \bar{f}(X_n)$, $d(\bar{f}(X), \bar{f}(Y)) \leq \tau d(X, Y)$ where $0 < \tau < 1$. Then the sequence X_n converges to a unique Fs fixed point.

Proof: By successive repeated iteration, we have

$$d(X_{n+1}, X_n) \leq \tau d(X_1, X_0)$$

$$\text{Since } 0 < \tau < 1$$



Because Fs space is complete: $\mathcal{X}_n \rightarrow \mathcal{X}^*$.

Taking limits $\bar{f}(\mathcal{X}^*) = \mathcal{X}^*$.

V. APPLICATIONS

A. Artificial Intelligence:

Fs-sets model:

- Multi-label classification
- Uncertain reasoning systems
- Rule-based inference engines

B. Machine learning:

- Used in future subset selection

C. Robotics:

Fs-set functions model

- Sensor fusion $S_{final} = S_1 \cup S_2 \cup \dots \cup S_n$
- Path uncertainty modelling

D. Optimization:

Fs sets define the search space: $\min f(G)$ where $A \in \mathcal{F}(X)$

Advanced Extensions:

FS-Dynamical Systems: $G_{t+1} = f_{\mathcal{F}}(G_t)$ used in iterative AI systems

Quantum-inspired FS-sets

Membership: $\mu(x) = \alpha|0\rangle + \beta|1\rangle$

E. Applications in Computational Mathematics:

The computational capabilities of Fs-subsets under Fs-set functions solve practical modelling problems where classical fuzzy sets fail

- Multi-Criteria Decision Making (MCDM):** Fs-sets can evaluate complex scenarios involving varying expert weights. The Fs-set function acts as a decision-aggregation operator mapping criteria sets ($\mathcal{U}$) to evaluation spaces ($\mathcal{V}$) using Boolean homomorphisms.
- Fuzzy soft Topology:** By employing images and inverse images of Fs-subsets, the structure can be extended to fuzzy soft topological spaces, facilitating algorithms for pattern recognition and medical diagnosis.

VI. CONCLUSION

The framework of Fs-subsets under Fs-set functions provides a robust mathematical environment that resolves limitations in traditional fuzzy mappings. By operating in a complete Boolean algebra, this framework ensures that fundamental algebraic laws (such as Fs-De Morgan laws) hold. Future work will apply this mathematical structure to advanced optimisation algorithms and uncertainty quantification.

ACKNOWLEDGEMENT

The authors are grateful to Dr V. Yogeswara and sincerely thank the anonymous reviewers and look forward to receiving valuable comments to improve the quality of the article.

DECLARATION STATEMENT

After aggregating input from all authors, I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organisations or agencies. This independence ensures that the research is conducted objectively and free from external influence.
- **Ethical Approval and Consent to Participate:** The data provided in this article is exempt from the requirement for ethical approval or participant consent.
- **Data Access Statement and Material Availability:** The data and materials in this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed equally to all participating individuals.

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