

On Solving Special Quadratic Diophantine Equations

R. Krishna, R. Sivaraman

Abstract: *Diophantine equations, named after the ancient Greek mathematician Diophantus, play a vital role not only in number theory but also in several branches of mathematics. In this paper, we solve a special quadratic Diophantine equation and derive its possible solutions.*

Keywords: *Quadratic Diophantine Equation, Positive integer solutions, Polar Form, Euler's Formula.*

I. INTRODUCTION

Diophantine equations are equations whose solutions must be integers. Since the solutions are integers, and often positive integers, such equations have more practical applications than other equations in mathematics. In this paper, first we will solve a special quadratic Diophantine equation. $Dx^2 + (D+1)y^2 = (2D+1)^n$ and provide its possible solutions. For recent methods of solving polynomial Diophantine equations, see [6].

A. Theorem 1

The positive integer solutions of the quadratic Diophantine equation

$$Dx^2 + (D+1)y^2 = (2D+1)^n \quad (1),$$

where D is a non-square number and n is an odd positive integer, are

$$x = \frac{(2D+1)^{n/2}}{\sqrt{D}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \right| \quad (2)$$

$$y = \frac{(2D+1)^{n/2}}{\sqrt{D+1}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \right| \quad (3)$$

PROOF:

First, for a given positive integer, we obtain the polar form of $\sqrt{D} + i\sqrt{D+1}$

$$\sqrt{D} + i\sqrt{D+1} = r e^{i\theta} = r (\cos \theta + i \sin \theta) \\ \Rightarrow r \cos \theta = \sqrt{D}, r \sin \theta = \sqrt{D+1} \quad (4)$$

From this, we obtain $r^2 = D + D + 1 = 2D + 1$

$$\Rightarrow r = (2D+1)^{1/2}, \theta = \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \quad (5)$$

Hence, the polar form of $(\sqrt{D} + i\sqrt{D+1})^n$ is given by

$$(\sqrt{D} + i\sqrt{D+1})^n = (2D+1)^{n/2} e^{in \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right)}$$

Now using Euler's formula, we obtain

$$(\sqrt{D} + i\sqrt{D+1})^n = (2D+1)^{n/2} \left\{ \cos \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] + i \sin \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \right\} \quad (6)$$

Assume that,

$$\sqrt{D} x + i\sqrt{D+1} y = (\sqrt{D} + i\sqrt{D+1})^n \quad (7)$$

$$\sqrt{D} x - i\sqrt{D+1} y = (\sqrt{D} - i\sqrt{D+1})^n \quad (8)$$

Now multiplying (7) & (8), we get,

$$Dx^2 + (D+1)y^2 = (2D+1)^n, \text{ which is (1).}$$

Equating the real & imaginary parts in (6) & (7), we get

$$x = \frac{(2D+1)^{n/2}}{\sqrt{D}} \cos \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \quad (9)$$

$$y = \frac{(2D+1)^{n/2}}{\sqrt{D+1}} \sin \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \quad (10)$$

Now, from (9) and (10), if we consider $(|x|, |y|)$ then these pairs would provide positive integer solutions to the given Quadratic Diophantine Equation $Dx^2 + (D+1)y^2 = (2D+1)^n$

Hence, the positive integer solutions to $Dx^2 + (D+1)y^2 = (2D+1)^n$ are given by

$$x = \frac{(2D+1)^{n/2}}{\sqrt{D}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \right| \quad (11)$$

$$y = \frac{(2D+1)^{n/2}}{\sqrt{D+1}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{D+1}{D}} \right) \right] \right| \quad (12)$$

We observe that if n is even, (11) and (12) do not provide integer solutions. This completes the proof.

We will now analyse some particular cases of Theorem 1.

CASE 1:

Let $D = 2$, n is any odd positive integer. Then the quadratic Diophantine equation is $2x^2 + 3y^2 = 5^n$ and its solutions from (11) & (12) are of the form

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$$x = \frac{5^{n/2}}{\sqrt{2}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{3}{2}} \right) \right] \right| \quad (13)$$

$$y = \frac{5^{n/2}}{\sqrt{3}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{3}{2}} \right) \right] \right| \quad (14)$$

For $n = 1, 3, 5, 7, 9, \dots$, the positive integer solutions of (x, y) to $2x^2 + 3y^2 = 5^n$ are given respectively by $(1, 1), (7, 3), (11, 31), (197, 13), (119, 801), \dots$

CASE 2:

Let $D = 17$, n is any odd positive integer.

Then the quadratic Diophantine equation is $17x^2 + 18y^2 = 35^n$ and its solutions from (11) & (12) are of the form

$$x = \frac{35^{n/2}}{\sqrt{17}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{18}{17}} \right) \right] \right| \quad (15)$$

$$y = \frac{35^{n/2}}{\sqrt{18}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{18}{17}} \right) \right] \right| \quad (16)$$

For $n = 1, 3, 5, 7, 9, \dots$, the positive integer solutions of (x, y) to $17x^2 + 18y^2 = 35^n$ are given respectively by $(1, 1), (37, 33), (1151, 1291), (47627, 37843), (1314721, 1657161), \dots$

CASE 3:

Let $D = 199$, n is any odd positive integer

Then the quadratic Diophantine equation is $199x^2 + 200y^2 = 399^n$ and its solutions from (11) & (12) are of the form

$$x = \frac{399^{n/2}}{\sqrt{199}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{200}{199}} \right) \right] \right| \quad (17)$$

$$y = \frac{399^{n/2}}{\sqrt{200}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{200}{199}} \right) \right] \right| \quad (18)$$

For $n = 1, 3, 5, 7, \dots$, the positive integer solutions of (x, y) to $199x^2 + 200y^2 = 399^n$ are given respectively by $(1, 1), (401, 397), (158399, 159995), (64156399, 62882807), \dots$

B. Theorem 2:

The positive integer solutions of the quadratic Diophantine equation

$Dx^2 + (D + 1)y^2 = (2D + 1)^n$, where $D = m^2$ is a square number and n is some odd positive integer, are

$$x = \frac{(2m^2 + 1)^{n/2}}{m} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right| \quad (19)$$

$$y = \frac{(2m^2 + 1)^{n/2}}{\sqrt{m^2 + 1}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right| \quad (20)$$

PROOF:

Consider $D = m^2$ for some integer m . Then we have $D + 1 = m^2 + 1$, $2D + 1 = 2m^2 + 1$

From equation (1), the original equation can be written as

$$m^2 x^2 + (m^2 + 1)y^2 = (2m^2 + 1)^n$$

$$\Rightarrow X^2 + (m^2 + 1)y^2 = (2m^2 + 1)^n, \text{ where } X = mx \Rightarrow$$

$$x = \frac{X}{m} \quad (21).$$

From (2) & (3) of Theorem 1, we will get the positive integer solutions as

$$x = \frac{(2m^2 + 1)^{n/2}}{m} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right|$$

$$y = \frac{(2m^2 + 1)^{n/2}}{\sqrt{m^2 + 1}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right|$$

Now, we will analyze some particular cases of Theorem 2.

CASE 1:

Let $m = 2$, n be some positive integer

$$\Rightarrow D = m^2 = 4$$

Then by (21) the original quadratic Diophantine equation is $X^2 + 5y^2 = 9^n$, where $X = 2x$.

$$\Rightarrow 4x^2 + 5y^2 = 9^n.$$

The positive integer solutions of the transformed equation $4x^2 + 5y^2 = 9^n$ are

$$x = \frac{9^{n/2}}{2} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right| = \frac{3^n}{2} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right|$$

$$y = \frac{9^{n/2}}{\sqrt{5}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right| = \frac{3^n}{\sqrt{5}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right|$$

For $n = 1, 3, 5, 7, \dots$, the positive integer solutions of (x, y) to $4x^2 + 5y^2 = 9^n$ are given by $(1, 1), (11, 7), (59, 95), (1009, 377), \dots$

If n is an even number, we have to solve the original equation $X^2 + 5y^2 = 9^n$ to get integer solutions. Using equations (19) and (20), we have

$$X = 3^n \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right|$$

$$y = \frac{3^n}{\sqrt{5}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{5}}{2} \right) \right] \right|$$

For $n = 2, 4, 6, 8, \dots$ the positive integer solutions of (X, y) to $X^2 + 5y^2 = 9^n$ are given by $(1, 4), (79, 8), (239, 308), (5921, 1264), \dots$

CASE 2:

Let $m = 6$, n be some positive integer

$$D = m^2 = 36$$

Using (21), the original quadratic Diophantine equation is $X^2 + 37y^2 = 73^n$, where $X = 6x$.

$$36x^2 + 37y^2 = 73^n$$

The positive integer solutions of the transformed equation $36x^2 + 37y^2 = 73^n$ are

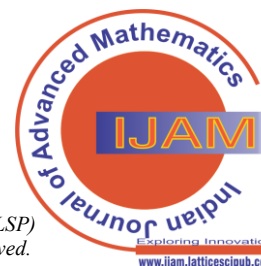
$$x = \frac{73^{n/2}}{6} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{37}}{6} \right) \right] \right|$$

$$y = \frac{73^{n/2}}{\sqrt{37}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{37}}{6} \right) \right] \right|$$

For $n = 1, 3, 5, 7, \dots$, the positive integer solutions of (x, y) to $36x^2 + 37y^2 = 73^n$ are given by $(1, 1), (75, 71), (5179, 5471), (4100833, 367417), \dots$

If n is an even number, we have to solve the original equation $X^2 + 37y^2 = 73^n$ to get integer solutions. Using equations (19) and (20), we have

$$X = 73^{n/2} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{37}}{6} \right) \right] \right|$$





$$y = \frac{73^{n/2}}{\sqrt{37}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{37}}{6} \right) \right] \right|$$

For $n = 2, 4, 6, 8, \dots$ the positive integer solutions of (X, y) to $X^2 + 37y^2 = 73^n$ are given by $(1, 12), (5327, 24), (15983, 63900), (28355617, 255696), \dots$

CASE 3:

Let $m = 9, n$ be some positive integer

$$D = m^2 = 81$$

By (21), the original quadratic Diophantine equation is $X^2 + 82y^2 = 163^n$, where $X = 9x$.

$$81x^2 + 82y^2 = 163^n$$

The positive integer solutions of the transformed equation $81x^2 + 82y^2 = 163^n$ are

$$x = \frac{163^{n/2}}{9} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{82}}{9} \right) \right] \right|$$

$$y = \frac{163^{n/2}}{\sqrt{82}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{82}}{9} \right) \right] \right|$$

For $n = 1, 3, 5, 7, \dots$, the positive integer solutions of (x, y) to $81x^2 + 82y^2 = 163^n$ are given by $(1, 1), (165, 161), (26239, 26891), (4436363, 4223827), \dots$

If n is an even number, we have to solve the original equation $X^2 + 82y^2 = 163^n$ to get integer solutions. Using equations (19) and (20), we have

$$X = 163^{n/2} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{82}}{9} \right) \right] \right|$$

$$y = \frac{163^{n/2}}{\sqrt{82}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{82}}{9} \right) \right] \right|$$

For $n = 2, 4, 6, 8, \dots$ the positive integer solutions of (X, y) to $X^2 + 82y^2 = 163^n$ are given by $(1, 18), (26567, 36), (79703, 478170), (705699217, 1912824), \dots$

CASE 4:

Let $m = 13, n$ be some positive integer

$$\Rightarrow D = m^2 = 169$$

Using (21), the original quadratic Diophantine equation is $X^2 + 170y^2 = 339^n$, where $X = 13x$.

$$\Rightarrow 169x^2 + 170y^2 = 339^n$$

The positive integer solutions of the transformed equation $169x^2 + 170y^2 = 339^n$ are

$$x = \frac{339^{n/2}}{13} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{170}}{13} \right) \right] \right|$$

$$y = \frac{339^{n/2}}{\sqrt{170}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{170}}{13} \right) \right] \right|$$

For $n = 1, 3, 5, 7, \dots$, the positive integer solutions of (x, y) to $169x^2 + 170y^2 = 339^n$ are given by $(1, 1), (341, 337), (114239, 115595), (39416539, 38497187), \dots$

If n is an even number, we have to solve the original equation $X^2 + 170y^2 = 339^n$ to get integer solutions. Using equations (19) and (20), we have

$$X = 339^{n/2} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{170}}{13} \right) \right] \right|$$

$$y = \frac{339^{n/2}}{\sqrt{170}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{170}}{13} \right) \right] \right|$$

For $n = 2, 4, 6, \dots$ the positive integer solutions of (X, y) to $X^2 + 170y^2 = 339^n$ are given by $(1, 26), (114919, 52), (344759, 2987842), \dots$

For more information about various methods of solving linear Diophantine Equations, solving Recurrence relations in connection with Diophantine Equations, and exponential Diophantine equations, see [1 – 5].

II. CONCLUSION

Considering the special quadratic Diophantine equation as in (1), we obtained the positive integer solutions in a novel way. In particular, in Theorem 1, we obtained the positive integer solutions of $Dx^2 + (D + 1)y^2 = (2D + 1)^n$, where D is non-square and n is an odd positive integer for the cases $D = 2, 17, 199$ using the equations

$$x = \frac{(2D + 1)^{n/2}}{\sqrt{D}} \left| \cos \left[n \tan^{-1} \left(\sqrt{\frac{D + 1}{D}} \right) \right] \right|$$

$$y = \frac{(2D + 1)^{n/2}}{\sqrt{D + 1}} \left| \sin \left[n \tan^{-1} \left(\sqrt{\frac{D + 1}{D}} \right) \right] \right|$$

respectively. Similarly, in Theorem 2, we obtained the positive integer solutions to the quadratic Diophantine equation $Dx^2 + (D + 1)y^2 = (2D + 1)^n$, where D is a square number and n is some positive integer for the cases $D = m^2 = 4, 36, 81, 169$ using the equations

$$x = \frac{(2m^2 + 1)^{n/2}}{m} \left| \cos \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right|$$

$$y = \frac{(2m^2 + 1)^{n/2}}{\sqrt{m^2 + 1}} \left| \sin \left[n \tan^{-1} \left(\frac{\sqrt{m^2 + 1}}{m} \right) \right] \right|$$

Thus, in this paper, we proved two theorems and verified some particular cases for D and n . Using these ideas, we can interestingly solve several quadratic Diophantine equations.

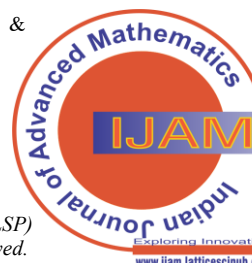
DECLARATION STATEMENT

After aggregating input from all authors, I, as the article's author, must verify the accuracy of the following information.

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