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A Family of Cotangent-Polylogarithm Integrals, Harmonic Sums, and Level-Four Cyclotomy

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Abstract: ***For integers*** $\nu \geq 2$ ***and parameters*** $0 \leq \lambda \leq 1$, ***we study the family*** $\mathcal{I}_\nu(\lambda) := \int_0^{\pi/2} \cot x \, \mathrm{Li}_\nu\left(\frac{\lambda}{1+\cot x}\right) dx$. ***We prove absolute convergence, reduce the integral to a Dirichlet series with positive coefficients, and evaluate it in terms of a depth-two multiple polylogarithm, with*** $\omega = (1+i)/2$: $\mathcal{I}_\nu(\lambda) = \frac{\pi}{2}\mathrm{Im}\mathrm{Li}_\nu(\lambda\omega) - \mathrm{Re}\mathrm{Li}_{\nu,1}(\lambda\omega, \omega^{-1})$. ***We then derive an exact second-order recurrence for the Dirichlet coefficients, an Euler differential descent in the deformation parameter, a functional-difference equation for the weight generating function, and an exponentially convergent expansion in generalized harmonic numbers and zeta values. As consequences, the sequence*** $\mathcal{I}_\nu := \mathcal{I}_\nu(1)$ ***is strictly decreasing,*** $\mathcal{I}_\nu = \frac{\pi}{4} + \left(\frac{\pi}{4} - \frac{1}{2}\right)2^{-\nu} + O(3^{-\nu})$, $\nu \to \infty$, ***and no nontrivial linear recurrence with constant coefficients in*** ν ***can hold. At*** $\lambda = 1$, ***the family gives a concrete one-dimensional model for level-four cyclotomic periods built from harmonic sums.***

Keywords: Polylogarithm, Multiple Polylogarithm, Cyclotomic Period, Harmonic Number, Euler Sum, Dirichlet Series.

I. INTRODUCTION AND STATEMENTS

The classical polylogarithm, its multiple variants, and their special values connect analysis, number theory, and mathematical physics. Panzer [1] develops identities and evaluations for multiple polylogarithms, while Xu and Wang [2] study the associated harmonic Euler sums. Au [9, 10] gives systematic methods for one-dimensional polylogarithmic integrals and iterated integrals at algebraic arguments. Related parametric integral evaluations appear in Sofo, Pain, and Scharaschkin [11]; Shalev [12] studies log-sine integrals and Apéry-like sums through polylogarithmic generating functions. These works provide context for the present family; the specific evaluation and coefficient identities below are proved directly.

For integers $\nu \geq 2$ and $0 \leq \lambda \leq 1$, define

$$\mathcal{I}_\nu(\lambda) := \int_0^{\pi/2} \cot x \, \mathrm{Li}_\nu\left(\frac{\lambda}{1+\cot x}\right) dx, \mathcal{I}_\nu := \mathcal{I}_\nu(1)$$

$$0 < \frac{\lambda}{1+\cot x} < 1 \qquad (0 < x < \pi/2,\ 0 < \lambda \leq 1),$$

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We use the principal branch of Li_ν on $\mathbb{C}\backslash[1,\infty)$, with the standard power-series normalization [13,§25.12(ii)]. Since no branch cut is crossed in (1). We also set

$$\omega := \frac{1+i}{2}, \bar{\omega} = \frac{1-i}{2}, \omega^{-1} = 1 - i, |\omega| = 2^{-\frac{1}{2}}. \qquad (2)$$

For complex α, β such that the series converges absolutely, our depth-two convention is

$$\mathrm{Li}_{\nu,1}(\alpha, \beta) := \sum_{\mu > \kappa \geq 1} \frac{\alpha^\mu \beta^\kappa}{\mu^\nu \kappa}. \qquad (3)$$

This is the nested-series convention for multiple polylogarithms in [10, §2.2]; see also [1, §1, Eq. (4)] for an alternative convention using increasing summation indices.

The main evaluation is as follows.

Theorem 1.

For every integer $\nu \geq 2$ and every $0 \leq \lambda \leq 1$, the integral $\mathcal{I}_\nu(\lambda)$ is absolutely convergent. Moreover,

$$\mathcal{I}_\nu(\lambda) = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu \lambda^\mu}{\mu^\nu}, \qquad (4)$$

where

$$\alpha_\mu := \int_0^1 \frac{\eta^{\mu-1}(1-\eta)}{1-2\eta+2\eta^2} d\eta > 0, \qquad \mu \geq 1, \qquad (5)$$

and

$$\mathcal{I}_\nu(\lambda) = \frac{\pi}{2}\mathrm{Im}\mathrm{Li}_\nu(\lambda\omega) - \mathrm{Re}\mathrm{Li}_{\nu,1}(\lambda\omega, \omega^{-1}). \qquad (6)$$

In particular,

$$\mathcal{I}_\nu = \frac{\pi}{2}\mathrm{Im}\mathrm{Li}_\nu(\omega) - \mathrm{Re}\mathrm{Li}_{\nu,1}(\omega, \omega^{-1}). \qquad (7)$$

The coefficients α_μ have a simple exact recurrence.

Theorem 2.

For every integer

$$\mu \geq 1, \alpha_\mu - 2\alpha_{\mu+1} + 2\alpha_{\mu+2} = \frac{1}{\mu(\mu+1)}. \qquad (8)$$

also

$$\alpha_1 = \frac{\pi}{4}, \qquad \alpha_2 = \frac{\pi}{4} - \frac{1}{2}, \qquad (9)$$

and

$$0 < \alpha_\mu \leq \frac{2}{\mu(\mu+1)} \qquad (\mu \geq 1). \qquad (10)$$

For the differential descent, extend (1) to $\nu = 1$ using $\mathrm{Li}_1(u) = -\log(1-u)$ and to $\nu = 0$ using $\mathrm{Li}_0(u) = u/(1-u)$. The corresponding integrals and series (1) converge for $0 \leq \lambda \leq 1$; at $(\nu, \lambda) = (0,1)$ the endpoint is understood by its finite limit. These extensions also follow from (10). Term-wise differentiation of the classical polylogarithm series [13, Eq. (25.12.10)] gives the following descent.

Theorem 3.

Let $\vartheta := \lambda\frac{d}{d\lambda}$.

For every integer $\nu \geq 1$

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and every $0 \le \lambda < 1$,

$$\vartheta \mathcal{I}_\nu(\lambda) = \mathcal{I}_{\nu-1}(\lambda), \qquad (11)$$

where

$$\mathcal{I}_0(\lambda) = \frac{\lambda\big(\pi + 2(1-\lambda)\log(1-\lambda)\big)}{2(\lambda^2 - 2\lambda + 2)}. \qquad (12)$$

Consequently,

$$\vartheta^\nu \mathcal{I}_\nu(\lambda) = \mathcal{I}_0(\lambda), \qquad (13)$$

and, for $0 < \lambda < 1$,

$$\mathcal{I}_\nu(\lambda) = \frac{1}{(\nu-1)!}\int_0^\lambda \frac{\mathcal{I}_0(\tau)}{\tau}\log^{\nu-1}\left(\frac{\lambda}{\tau}\right) d\tau. \qquad (14)$$

At $\lambda = 0$, the right-hand side is defined by continuity and equals zero.

The weight generating function satisfies a functional-difference equation.

Proposition 4.

Let $\Phi(\sigma) := \sum_{\nu=1}^{\infty} \mathcal{I}_\nu \sigma^{\nu-1} \qquad (|\sigma| < 1). \qquad (15)$

Then

$$\Phi(\sigma) = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu}{\mu - \sigma}, \qquad (16)$$

which extends meromorphically to $\mathbb{C}$, with simple poles at 1,2,3, ... Moreover, as an identity of meromorphic functions,

$$\Phi(\sigma) - 2\Phi(\sigma+1) + 2\Phi(\sigma+2) = \frac{1-\gamma-\psi(1-\sigma)}{\sigma(\sigma+1)} - \frac{\pi}{2(\sigma+1)}, \qquad (17)$$

where ψ is the digamma function and γ is Euler's constant.

The connection between nested polylogarithmic sums and harmonic Euler sums is classical [1, 2]. The depth-two term in (7) admits the following explicit harmonic expansion, derived in Section III.

Theorem 5.

For every integer $\nu \ge 2$,

$$\mathcal{I}_\nu = \frac{\pi}{2}\mathrm{Im}\mathrm{Li}_\nu(\omega) + \sum_{j=2}^{\nu} \zeta(j)\mathrm{Re}\mathrm{Li}_{\nu+1-j}(\omega) - \sum_{j=1}^{\nu}\sum_{\rho=1}^{\infty} \frac{H_\rho^{(j)}\mathrm{Re}(\omega^\rho)}{\rho^{\nu+1-j}}, \qquad (18)$$

where

$$H_\rho^{(j)} := \sum_{m=1}^{\rho} \frac{1}{m^j}, \qquad H_\rho^{(1)} = H_\rho.$$

since

$$\mathrm{Re}(\omega^\rho) = 2^{-\rho/2}\cos\left(\frac{\pi\rho}{4}\right), \qquad (19)$$

Formula (18) is exponentially convergent.

A final consequence concerns monotonicity and asymptotics in the weight.

Corollary 6.

The Dirichlet series

$$\mathcal{D}(s) := \sum_{\mu=1}^{\infty} \frac{\alpha_\mu}{\mu^s} \qquad (20)$$

converges absolutely for $\mathrm{Re}\, s > -1$, is holomorphic there, and satisfies

$$(-1)^r \frac{d^r}{ds^r}\mathcal{D}(s) > 0 \qquad (s > -1,\ r = 0,1,2,\dots).$$

In particular, the sequence $\mathcal{I}_\nu = \mathcal{D}(\nu)$ is strictly decreasing, $\mathcal{I}_\nu \downarrow \pi/4$, and

$$\mathcal{I}_\nu = \frac{\pi}{4} + \left(\frac{\pi}{4} - \frac{1}{2}\right)2^{-\nu} + O(3^{-\nu}). \qquad (21)$$

Moreover, no nontrivial linear recurrence with constant coefficients in ν can hold for the sequence $(\mathcal{I}_\nu)_{\nu \ge 2}$.

Remark 7.

The cyclotomic interpretation applies to the specialization $\lambda = 1$. In the unit-interval variable η, the relevant singular points are $0, 1, \omega, \bar{\omega}, \infty$, since

$$1 - 2\eta + 2\eta^2 = (1 - \eta/\omega)(1 - \eta/\bar{\omega}).$$

The Möbius transformation $\tau = \eta/(1-\eta)$ sends these points to 0, ∞, i, −i, −1, respectively, a subset of the level-four alphabet {0, 1, −1, i, −i, ∞}. Thus, the iterated integrals for $\mathcal{I}_\nu(1)$ have a level-four cyclotomic description. The invariance under Möbius changes of variables and the identification with cyclotomic multiple zeta values are made precise in [10, Proposition 3.2(4), Theorem 3.5, and Corollary 3.6].

For the underlying cyclotomic theory, see Goncharov [3, §§1.1 and 2.8]; for double-shuffle and other standard relations at roots of unity, see Yaddaden [4] and Bachmann and Yaddaden [6]. Brown provides the broader motivic framework [5, §10.6.2], while Ablinger and Schneider [7] study cyclotomic harmonic sums, and Ablinger, Blümlein, Round, and Schneider [8, §2] review cyclotomic polylogarithms. These references concern the structural setting, not the particular evaluation (6).

For $0 < \lambda < 1$, $\mathrm{Li}_\nu(\lambda\eta)$ introduces the additional point $\eta = 1/\lambda$, which maps to $\tau = 1/(\lambda - 1)$. Consequently, the same argument does not establish a level-four period statement for arbitrary λ; the general family remains polylogarithmic, with a parameter-dependent alphabet.

II. PROOF OF THE EVALUATION

We begin with convergence and the reduction to the unit interval.

Lemma 8.

For $\nu \ge 2$ and $0 \le \lambda \le 1$,

$$\mathcal{I}_\nu(\lambda) = \int_0^1 \mathrm{Li}_\nu(\lambda\eta)\frac{1-\eta}{\eta(1-2\eta+2\eta^2)}\, d\eta. \qquad (22)$$

The integral is absolutely convergent.

Proof. As $x \to 0^+$,

$$\cot x = \frac{1}{x} + O(x), \qquad \frac{\lambda}{1+\cot x} = \lambda x + O(x^2),$$
$$\mathrm{Li}_\nu(u) = u + O(u^2),$$

So the integrand in (1) equals $\lambda + O(x)$. As $x \to \pi/2^-$, if $u := \pi/2 - x$, then

$$\cot x = u + O(u^3), \qquad \frac{\lambda}{1+\cot x} = \lambda + O(u),$$

and $\mathrm{Li}_\nu(\lambda + O(u))$ remains bounded for $0 \le \lambda \le 1$, so the integrand is $O(u)$. Hence the integral is absolutely convergent. Now set $\tau = \tan x$ and then $\eta = \tau/(1+\tau)$.

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Since
$\frac{\lambda}{1+\cot x} = \frac{\lambda\tau}{1+\tau} = \lambda\eta, \qquad dx = \frac{d\tau}{1+\tau^2} = \frac{d\eta}{1-2\eta+2\eta^2},$
and $\cot x\, dx = \frac{1-\eta}{\eta(1-2\eta+2\eta^2)}\, d\eta,$
formula (22) follows.

Lemma 9.

For $\nu \geq 2$ and $0 \leq \lambda \leq 1$,

$$\mathcal{I}_\nu(\lambda) = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu \lambda^\mu}{\mu^\nu},$$

$$\alpha_\mu = \int_0^1 \frac{\eta^{\mu-1}(1-\eta)}{1-2\eta+2\eta^2}\, d\eta.$$

In particular, each α_μ is positive.

Proof. For $0 \leq \eta \leq 1$ and $0 \leq \lambda \leq 1$, the defining polylogarithm series [13, Eq. (25.12.10)]

$$\mathrm{Li}_\nu(\lambda\eta) = \sum_{\mu=1}^{\infty} \frac{(\lambda\eta)^\mu}{\mu^\nu}$$

has nonnegative terms. By Tonelli's theorem and Lemma 8,

$$\mathcal{I}_\nu(\lambda) = \sum_{\mu=1}^{\infty} \frac{\lambda^\mu}{\mu^\nu} \int_0^1 \frac{\eta^{\mu-1}(1-\eta)}{1-2\eta+2\eta^2}\, d\eta,$$

which is the claim.

Lemma 10.

For every integer $\mu \geq 1$,

$$\alpha_\mu = \frac{\pi}{2}\mathrm{Im}(\omega^\mu) - \mathrm{Re}\sum_{\kappa=1}^{\mu-1} \frac{\omega^{\mu-\kappa}}{\kappa}. \qquad (23)$$

Proof. Because

$$1-2\eta+2\eta^2 = (1-\eta/\omega)(1-\eta/\bar{\omega}),$$

a direct partial fraction decomposition gives

$$\frac{1-\eta}{\eta(1-2\eta+2\eta^2)} = \frac{1}{\eta} + \frac{\bar{\omega}}{1-\eta/\omega} + \frac{\omega}{1-\eta/\bar{\omega}}. \qquad (24)$$

Hence

$$\alpha_\mu = \frac{1}{\mu} + \bar{\omega}\mathrm{J}_\mu(\omega^{-1}) + \omega\mathrm{J}_\mu(\bar{\omega}^{-1}) \qquad (24a)$$

where

$$\mathrm{J}_\mu(\alpha) := \int_0^1 \frac{\eta^\mu}{1-\alpha\eta}\, d\eta.$$

If $\alpha \notin [1,\infty)$ and $\alpha \neq 0$, the segment from 0 to α does not cross the principal logarithmic cut, and the substitution $u = \alpha\eta$ yields

$$\mathrm{J}_\mu(\alpha) = \alpha^{-\mu-1} \int_0^\alpha \frac{u^\mu}{1-u}\, du.$$

Using

$$\frac{u^\mu}{1-u} = \frac{1}{1-u} - \sum_{\kappa=0}^{\mu-1} u^\kappa,$$

we obtain

$$\mathrm{J}_\mu(\alpha) = -\alpha^{-\mu-1}\left(\log(1-\alpha) + \sum_{\kappa=1}^{\mu} \frac{\alpha^\kappa}{\kappa}\right). \qquad (25)$$

Now

$$1-\omega^{-1} = i, \qquad 1-\bar{\omega}^{-1} = -i, \qquad \log(i) = \frac{i\pi}{2}, \qquad \log(-i) = -\frac{i\pi}{2},$$

with the principal logarithm. Substituting (25) into (24a), using $\bar{\omega}\omega^{\mu+1} = \omega^\mu/2$ and $\omega\bar{\omega}^{\mu+1} = \bar{\omega}^\mu/2$, and then cancelling the term $1/\mu$, we get

$$\alpha_\mu = \frac{\pi}{2}\mathrm{Im}(\omega^\mu) - \mathrm{Re}\sum_{\kappa=1}^{\mu-1} \frac{\omega^{\mu-\kappa}}{\kappa},$$

as required.

Proof of Theorem 1. Lemma 8 gives absolute convergence, and Lemma9 gives (4). Multiply (23) by λ^μ/μ^ν and sum over $\mu \geq 1$. The first term gives

$$\sum_{\mu=1}^{\infty} \frac{\lambda^\mu}{\mu^\nu}\frac{\pi}{2}\mathrm{Im}(\omega^\mu) = \frac{\pi}{2}\mathrm{Im}\sum_{\mu=1}^{\infty} \frac{(\lambda\omega)^\mu}{\mu^\nu} = \frac{\pi}{2}\mathrm{Im}\mathrm{Li}_\nu(\lambda\omega).$$

For the second term we write $\rho = \mu - \kappa \geq 1$ and estimate

$$\sum_{\mu>\kappa\geq 1} \frac{|\lambda\omega|^\mu |\omega|^{-\kappa}}{\mu^\nu \kappa} = \sum_{\rho=1}^{\infty} |\omega|^\rho \sum_{\kappa=1}^{\infty} \frac{\lambda^{\rho+\kappa}}{(\rho+\kappa)^\nu \kappa}.$$

The inner sum is

$$\ll \rho^{-\nu}\sum_{\kappa\leq\rho} \frac{1}{\kappa} + \sum_{\kappa>\rho} \frac{1}{\kappa^{\nu+1}} \ll \rho^{-\nu}\log(\rho+1),$$

So the outer sum converges because $|\omega| < 1$. Therefore, the double series converges absolutely, and we may rearrange:

$$\sum_{\mu=1}^{\infty} \frac{\lambda^\mu}{\mu^\nu}\mathrm{Re}\sum_{\kappa=1}^{\mu-1} \frac{\omega^{\mu-\kappa}}{\kappa} = \mathrm{Re}\sum_{\mu>\kappa\geq 1} \frac{(\lambda\omega)^\mu \omega^{-\kappa}}{\mu^\nu \kappa}$$
$$= \mathrm{Re}\mathrm{Li}_{\nu,1}(\lambda\omega, \omega^{-1}).$$

This proves (6). The special case (7) is obtained by setting $\lambda = 1$.

III. STRUCTURAL CONSEQUENCES

Proof of Theorem 2. The bound follows from

$$1-2\eta+2\eta^2 = 2\left(\eta-\frac{1}{2}\right)^2 + \frac{1}{2} \geq \frac{1}{2}, \qquad 0 \leq \eta \leq 1,$$

which gives

$$0 < \alpha_\mu \leq 2\int_0^1 \eta^{\mu-1}(1-\eta)\, d\eta = \frac{2}{\mu(\mu+1)}.$$

also,

$$\alpha_\mu - 2\alpha_{\mu+1} + 2\alpha_{\mu+2} = \int_0^1 \eta^{\mu-1}(1-\eta)\, d\eta = \frac{1}{\mu(\mu+1)},$$

which is (8). Finally,

$$\alpha_1 = \int_0^1 \frac{1-\eta}{1-2\eta+2\eta^2}\, d\eta = \frac{\pi}{4},$$

$$\alpha_2 = \int_0^1 \frac{\eta(1-\eta)}{1-2\eta+2\eta^2}\, d\eta = \frac{\pi}{4} - \frac{1}{2}.$$

Proof of Theorem 3. For $0 \leq \lambda < 1$, the series in (4) converges locally uniformly in λ; hence termwise differentiation is valid and gives

$$\vartheta\mathcal{I}_\nu(\lambda) = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu \lambda^\mu}{\mu^{\nu-1}} = \mathcal{I}_{\nu-1}(\lambda).$$

To compute $\mathcal{I}_0(\lambda)$, start from Lemma 8 with $\mathrm{Li}_0(u) = u/(1-u)$:

$$\mathcal{I}_0(\lambda) = \int_0^1 \frac{\lambda(1-\eta)}{(1-\lambda\eta)(1-2\eta+2\eta^2)}\, d\eta.$$

A direct decomposition yields

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$$\frac{\lambda(1-\eta)}{(1-\lambda\eta)(1-2\eta+2\eta^2)} = \frac{\lambda^2(\lambda-1)}{\lambda^2-2\lambda+2}\frac{1}{1-\lambda\eta} + \frac{\lambda(\lambda-1)}{\lambda^2-2\lambda+2}\frac{2\eta-1}{1-2\eta+2\eta^2} + \frac{\lambda}{\lambda^2-2\lambda+2}\frac{1}{1-2\eta+2\eta^2}$$

The middle term integrates to zero, while

$$\int_0^1 \frac{d\eta}{1-2\eta+2\eta^2} = \frac{\pi}{2}.$$

This gives (12). Formula (13) follows by iteration. Finally, (14) is the unique solution of (11) with $\mathcal{I}_\nu(0) = 0$, obtained by integrating the first-order equation repeatedly.

Proof of Proposition 4. From (10), we have $\alpha_\mu = O(\mu^{-2})$, so the series in (16) converges locally uniformly on compact subsets of $\mathbb{C}\backslash\{1,2,3,\dots\}$. For $|\sigma| < 1$,

$$\sum_{\nu=1}^{\infty} \frac{\sigma^{\nu-1}}{\mu^\nu} = \frac{1}{\mu-\sigma},$$

and therefore

$$\Phi(\sigma) = \sum_{\mu=1}^{\infty} \alpha_\mu \sum_{\nu=1}^{\infty} \frac{\sigma^{\nu-1}}{\mu^\nu} = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu}{\mu-\sigma}.$$

Next,

$$\Phi(\sigma+1) = -\frac{\alpha_1}{\sigma} + \sum_{\mu=1}^{\infty} \frac{\alpha_{\mu+1}}{\mu-\sigma}, \Phi(\sigma+2) = -\frac{\alpha_1}{\sigma+1} - \frac{\alpha_2}{\sigma} + \sum_{\mu=1}^{\infty} \frac{\alpha_{\mu+2}}{\mu-\sigma}.$$

Using Theorem 2,

$$\Phi(\sigma) - 2\Phi(\sigma+1) + 2\Phi(\sigma+2) = \sum_{\mu=1}^{\infty} \frac{1}{\mu(\mu+1)(\mu-\sigma)} + \frac{1}{\sigma} - \frac{\pi}{2(\sigma+1)}.$$

Now

$$\frac{1}{\mu(\mu+1)(\mu-\sigma)} = \frac{1}{\sigma(\sigma+1)}\left(\frac{1}{\mu-\sigma} - \frac{1}{\mu}\right) - \frac{1}{\sigma+1}\left(\frac{1}{\mu} - \frac{1}{\mu+1}\right),$$

so the classical digamma expansion [13, Eq. (5.7.6)]

$$-\psi(1-\sigma) - \gamma = \sum_{\mu=1}^{\infty}\left(\frac{1}{\mu-\sigma} - \frac{1}{\mu}\right)$$

yields

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu(\mu+1)(\mu-\sigma)} = \frac{-\psi(1-\sigma)-\gamma}{\sigma(\sigma+1)} - \frac{1}{\sigma+1}.$$

Substitution proves (17).

Proof of Theorem 5. The reduction below uses the harmonic-sum language familiar from the study of Euler sums; see Xu and Wang [2] for that broader context. Start from (7) and write $\rho = \mu - \kappa$ in the depth-two series:

$$\mathrm{Li}_{\nu,1}(\omega,\omega^{-1}) = \sum_{\rho=1}^{\infty} \omega^\rho \sum_{\kappa=1}^{\infty} \frac{1}{\kappa(\kappa+\rho)^\nu}.$$

For fixed $\rho \geq 1$ and $\nu \geq 2$, a direct partial fraction identity gives

$$\frac{1}{\kappa(\kappa+\rho)^\nu} = \frac{1}{\rho^\nu}\left(\frac{1}{\kappa} - \frac{1}{\kappa+\rho}\right) - \sum_{j=2}^{\nu} \frac{1}{\rho^{\nu+1-j}(\kappa+\rho)^j}. \qquad (26)$$

Summing over $\kappa \geq 1$, we obtain

$$\sum_{\kappa=1}^{\infty} \frac{1}{\kappa(\kappa+\rho)^\nu} = \frac{H_\rho}{\rho^\nu} - \sum_{j=2}^{\nu} \frac{\zeta(j) - H_\rho^{(j)}}{\rho^{\nu+1-j}}.$$

Therefore

$$\mathrm{Li}_{\nu,1}(\omega,\omega^{-1}) = \sum_{\rho=1}^{\infty} \omega^\rho \left(\frac{H_\rho}{\rho^\nu} - \sum_{j=2}^{\nu} \frac{\zeta(j) - H_\rho^{(j)}}{\rho^{\nu+1-j}}\right).$$

Take real parts and use

$$\sum_{\rho=1}^{\infty} \frac{\mathrm{Re}(\omega^\rho)}{\rho^s} = \mathrm{ReLi}_s(\omega), \qquad \mathrm{Re}(\omega^\rho) = 2^{-\rho/2}\cos\left(\frac{\pi\rho}{4}\right),$$

which is (19). Formula (18) follows.

Proof of Corollary 6. Because of (10), the series (20) converges absolutely for $\mathrm{Re}\, s > -1$ and defines a holomorphic function there. For real $s > -1$ and integers $r \geq 0$, termwise differentiation gives

$$(-1)^r \frac{d^r}{ds^r}\mathcal{D}(s) = \sum_{\mu=1}^{\infty} \frac{\alpha_\mu (\log\mu)^r}{\mu^s} > 0,$$

so $\mathcal{D}$ is completely monotone on $(-1,\infty)$. In particular, $\mathcal{I}_{\nu+1} < \mathcal{I}_\nu$.

From Theorem 2,

$$\mathcal{I}_\nu = \alpha_1 + \alpha_2 2^{-\nu} + \sum_{\mu=3}^{\infty} \frac{\alpha_\mu}{\mu^\nu} = \frac{\pi}{4} + \left(\frac{\pi}{4} - \frac{1}{2}\right)2^{-\nu} + R_\nu.$$

Using (10),

$$0 < R_\nu \leq 3^{-\nu} \sum_{\mu=3}^{\infty} \alpha_\mu \leq 2 \cdot 3^{-\nu} \sum_{\mu=3}^{\infty} \frac{1}{\mu(\mu+1)} = \frac{2}{3} 3^{-\nu},$$

which proves (21) and $\mathcal{I}_\nu \to \alpha_1 = \pi/4$.

Finally, assume that constants $c_0, \dots, c_r$ satisfy

$$\sum_{j=0}^{r} c_j \mathcal{I}_{\nu+j} = 0 \qquad \text{for all large } \nu.$$

Using (4) with $\lambda = 1$, we obtain

$$\sum_{\mu=1}^{\infty} \alpha_\mu \mu^{-\nu}\left(\sum_{j=0}^{r} c_j \mu^{-j}\right) = 0 \qquad \text{for all large } \nu.$$

Set $P(z) := \sum_{j=0}^{r} c_j z^j$. If some coefficient $a_\mu := \alpha_\mu P(1/\mu)$ is nonzero, let m be the least such index. Multiplication by m^ν gives

$$0 = a_m + \sum_{\mu>m} a_\mu \left(\frac{m}{\mu}\right)^\nu.$$

Since P is bounded on $[0,1]$ and $\sum_\mu \alpha_\mu < \infty$ by (10), the sequence (a_μ) is absolutely summable. Dominated convergence as $\nu \to \infty$ therefore gives $a_m = 0$, a contradiction. Thus, using $\alpha_\mu > 0$,

$$\sum_{j=0}^{r} c_j \mu^{-j} = 0 \qquad \text{for every } \mu \geq 1.$$

A nonzero polynomial in $1/\mu$ cannot vanish at infinitely many points, so all c_j vanish.

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Remark 11.

The involution $\tau \mapsto \tau^{-1}$ in the form

$$\mathcal{I}_\nu = \int_0^\infty \frac{\mathrm{Li}_\nu\left(\frac{\tau}{1+\tau}\right)}{\tau(1+\tau^2)}\,d\tau$$

gives the reciprocal identity

$$\mathcal{I}_\nu = \int_0^\infty \frac{\tau \mathrm{Li}_\nu\left(\frac{1}{1+\tau}\right)}{1+\tau^2}\,d\tau.$$

This symmetry is elementary, but it fits well with the relation $\omega/(1-\omega) = i$ behind the level-four factorization. The factors $1+\tau$ and $1+\tau^2$ are the cyclotomic polynomials $\Phi_2(\tau)$ and $\Phi_4(\tau)$, respectively; their rational kernels are part of the cyclotomic-polylogarithm alphabet in [8, §2, Eqs. (2.1)–(2.2)].

IV. CONCLUSION

We obtained exact polylogarithmic representations of the cotangent integral family. Its Dirichlet coefficients yield recurrence, differential, and generating-function identities. At $\lambda = 1$, an exponentially convergent harmonic expansion complements the weight asymptotics. The integral values decrease to $\pi/4$ and satisfy no nontrivial constant-coefficient linear recurrence in the weight. This specialization admits a level-four cyclotomic interpretation. For $0 < \lambda < 1$, the argument does not establish the same interpretation.

DECLARATION STATEMENT

After aggregating input from all authors, I must verify the accuracy of the following information as the article's author.

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