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One Integer Point on Some Elliptic Curve to Have Rank 1

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Abstract: ***Walsh proved a necessary condition for some curves of the form $y^2 = x^3 - px$, with p is a prime, to have Mordell - Weil rank 1. In this paper, we extend Walsh's result to show that some curves have Mordell-Weil rank 1.***

Keywords. Elliptic Curves, Integer Points, Rank of Elliptic Curve, Mordell-Weil Rank, and Diophantine Equation.

I. INTRODUCTION

Many authors have studied the relationship between an elliptic curve's rank and the size of its set of Integer points. Silverman [3] showed that the number of integer solutions to a cubic Thue equation $f(x, y) = m$ is bounded in terms of the rank of the Mordell - Weil group of the corresponding cubic curve $f(x, y) = mz^3$.

For a curve of the form

$$E: y^2 = x^3 - px$$

With p prime, it is known that the rank of the Mordell-Weil group is at most 2, which can be proved using the 2-descent methods in Silverman [3]. H. Daghigh & S. Didari [1] studied the rank of E: $y^2 = x^3 = pqx$. In particular, Walsh [2] gave a general condition for the rank by considering the number of integer points on E. Recently, Mina and Bacani [6] determined necessary conditions for the rank of the curve $y^2 = x^3 - 3pqx$ to be zero.

II. PRELIMINARIES

In this section, we introduce the objects we need later and collect some important facts about them. Let E be an elliptic curve defined by

$$E: y^2 = x^3 - px$$

With n integer. And as in [6], we define

$$\tilde{E}: y^2 = x^3 + 4px.$$

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To compute the rank (E) as in [6], we study the solvability of a Diophantine equation and rely on the methods of Sankari and Abdo [4], [5].

Let p is a prime and $n = 1,2,3$. Let

$$E_n/\mathbb{Q}: y^2 = x^3 - p^n x$$

Proposition 2.1. (See [3] or [7]). Let $E_n: y^2 = x^3 - p^n x$ be an elliptic curve, $\text{rank}E_n(\mathbb{Q}) = 1$ if and only if exactly one of the equations

$$-S^4 + p^n T^4 = U^2 \quad (1)$$
$$2S^4 + 2p^n T^4 = U^2 \quad (2)$$

Has a primitive solution, such that $\gcd(S,T) = \gcd(S,U) = \gcd(T,U) = 1$.

III. NEW RESULTS

Our principal result is the following. In the following Theorems, we consider the integer points on some curves to have Mordell-Weil rank 1. Theorem 3. 1. Let $n = 1, u$ be odd, v be even, $r > 1$ be an integer and p is a prime number.

i. If $(-1,2u) \in E_1(\mathbb{Q})$, then $p = 4u^2 + 1$ and $\text{rank}(E_1) = 1$.

ii. If $\left(v^2, v(v^2-1)\right) \in E_1(\mathbb{Q})$, then $p = 2v^2 - 1$ and $\text{rank}(E_1) = 1$.

iii. If $\left(pt^2, \frac{pr(r^2+3)t}{2}\right) \in E_1(\mathbb{Q})$ such that norm $(r + \sqrt{2}t) = -1$ and $p \equiv 5(\text{mod}8)$, then $p = r^2 + 4$ and $\text{rank}(E_1(\mathbb{Q})) = 1$.

Proof. i. We have $(-1,2u) \in E_1(\mathbb{Q})$ we easily find that $p = 4u^2 + 1$ and $p(1)^4 - (1)^4 = (2u)^2$, therefore the equation (1) in Proposition 2.1 has a primitive solution. And it is easy to see that the equation (2) in Proposition Therefore, the equation (2) in Proposition2.1 does not have a primitive solution (by mod 4). It follows from Proposition 2.1 that $\text{rank}(E_1) = 1$.

ii. we have $\left(v^2, v(v^2-1)\right) \in E_1(\mathbb{Q})$ we easily find that $p = 2v^2 - 1$ and $2p(1)^4 + 2(1)^4 = (2m)^2$, therefore the equation (2) in Proposition 2.1 does not have a primitive solution. It is easy to see that equation (1) in Proposition 2.1 has a primitive solution (by mod 4). It follows from Proposition 2.1 that $\text{rank}(E_1) = 1$.

iii. we have $\left(pt^2, \frac{pr(r^2+3)t}{2}\right) \in E_1(\mathbb{Q})$ such that norm $(r + \sqrt{2}t) = -1$, we easily find that $p = r^2 + 4$ and $p \equiv 5$ (mod 8), therefor the equation (1) in Proposition 2.1 has a primitive solution (this solution is $(1,1,2m)$). And it is easy to see that the equation (2) in Proposition 2.1 does not have a primitive solution (mod 8). It follows

from Proposition 2.1 that rank $(E_1) = 1$.

Theorem 3. 2. Let $n = 2, v$ be even and p is a prime number.

If $(v^2, v(v^2-1)) \in E_1(\mathbb{Q})$ and $p \equiv -1(\text{mod}8)$,then $p^2 = 2v^2 - 1$ and $\text{rank}(E_1) = 1$.

proof. we have $(v^2, v(v^2-1)) \in E_1(\mathbb{Q})$ We easily find that $p = 2v^2 - 1$ and $2p^2(1)^4 + 2(1)^4 = (2m)^2$ Therefore, the equation (2) in Proposition 2.1 has a primitive solution. It is easy to see that the equation (1) in Proposition 2.1 does not have a primitive solution (by mod 8). It follows from Proposition 2.1 that $\text{rank}(E_1) = 1$.

Theorem 3. 3. Let $n = 3, u, v$ are integers and p is a prime number.

If $(-v^2, uv) \in E_1(\mathbb{Q})$ and $p \equiv 5(\text{mod}8)$,then $p^3 = u^2 + v^4$ and $\text{rank}(E_1) = 1$.

proof. we have $(-v^2, uv) \in E_1(\mathbb{Q})$ We easily find that $p^3 = u^2 + v^4$ and $p^3(1)^4 - (b)^4 = (a)^2$, therefore the equation (1) in Proposition 2.1 has a primitive solution. It is easy to see that equation (2) in Proposition 2.1 does not have a primitive solution (by mod 8). It follows from Proposition 2.1 that $\text{rank}(E_1) = 1$.

IV. CONCLUSION

By extending Walsh's theorem to curves with Mordell-Weil rank 1, this study fills a critical gap in the literature. It provides a more comprehensive framework for understanding the arithmetic properties of these algebraic structures.

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DECLARATION STATEMENT

Some of the cited references are older and are noted explicitly as [1], [3] and [7]. However, these works remain significant for the current study, as they are pioneering in their fields.

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