Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-6 Issue-1, April 2026

Fractional form of Jensen's Inequality for the Exponential Integral Function with Applications to Modeling Utility

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***Abstract:** Jensen's inequality is a fundamental result in probability and analysis, that offers simple bounds for the convex function applied to a random variable. However, the classical form of this process does not include memory effects, which are very important in some physical and financial systems with long-range dependence. In this paper, we introduce a fractional-order generalisation of Jensen's inequality involving memory effects that can be accounted for by means of fractional calculus. We concentrate on the exponential integral function Ei(x) because of its wide use. For a random variable X with mean μ supported on $[1, \infty)$, and for every fractional-order $\alpha \in (0, 1)$ we show the strict inequality $Ei(\mu) \leq E[Ei(X)] - M(\alpha)$, where the new quantity $M(\alpha)$ is a memory correction defined in terms of Riemann–Liouville fractional integrals of order $1-\alpha$ of the function e^t/t. The correction term provides a whole family of bounds, controlled by the parameter $\alpha \in (0, 1]$, and depending on the specific behavior of X, as $\alpha \to 0^+$, the bound reduces to Jensen's gap, while for $\alpha \to 1^-$, the right-hand side approaches a new non-zero and path-dependent bound. The inequality is strict for non-degenerate X. (When $M(\alpha) \geq 0$, the bound is two-sided, $Ei(\mu) \leq E[Ei(X)] - M(\alpha) \leq E[Ei(X)]$.) The proof is based on order-monotonicity properties of fractional integrals. An equivalent formulation of the result in terms of Caputo derivatives is also given. An illustrative interpretation is discussed in the context of economic utility, where the resulting bounds may be viewed as capturing nonlocal averaging effects in convex (risk-seeking) utility evaluations.*

***Keywords:** Jensen's Inequality, Fractional Calculus, Caputo Derivative, Exponential Integral, Convex Analysis, Recursive Utility, Memory Effects, Risk Assessment*

I. INTRODUCTION

Jensen's inequality is a fundamental result in convex analysis and probability theory that relates the expectation of a convex function to the value of the function at the expectation. The inequality is particularly useful for understanding the relationship between the expected value of a function applied to an estimator and the function of the estimator's expected value [1].

Due to its generality, it has found wide-ranging applications across disciplines such as economics, statistics, and decision theory [2]. Jensen's inequality is a very important tool in the analysis of risk aversion and utility functions in economics, as it helps to describe the influence of uncertainty on the expected utility [3]. Also, in information theory, Jensen's inequality is a very important tool for analysing entropy and divergence functions, providing a theoretical foundation for deriving fundamental inequalities for comparing probability distributions and bounding the representation and transmission of information [4]. At the same time, Jensen's inequality is widely used in finance to analyse asset pricing, portfolio returns, and the pricing of financial options and derivatives under uncertainty. It was notably used by [5] to derive upper bounds on the valuation of debt in coupled financial networks, such as the Eisenberg-Noe models with bankruptcy costs, which provide theoretical guarantees for debt pricing under systemic risk.

Apart from the traditional form, Jensen-type inequalities have been refined substantially in recent years. Certain research has focused on obtaining more precise bounds using generalised convexity and fractional operators, covering a wide range of power mean inequalities, Hölder-type inequalities, and generalised estimates of f-divergences in information theory [6, 7]. For example, Abbaszadeh et al. [8] generalized complete and generalized Jensen inequalities using Riemann-Liouville fractional integrals, obtaining fractional generalizations of Hölder's and Minkowski's inequalities. Jarad et al [9] established a stochastic fractional integral framework, which they used to derive Jensen Mercer type inequalities that included fractional-order effects in their expectation structures. Likewise, Hyder et al. [10] generalized Hermite-Hadamard, trapezoid, and midpoint inequalities via fractional extended Riemann-Liouville integrals, focusing on analytical identities and graphical verification. Sadek et al. [11] generalized it to generalized convexity on fractal sets, obtaining extended Jensen and Hermite-Hadamard inequalities with applications in a wide range of fields. Bosch et al. refined the inequality for harmonic convex functions and generalised them to Caputo-type fractional integrals and local fractional derivatives. Habibzadeh-Omam et al. [12] further introduced fractional

Manuscript received on 07 January 2026 | First Revised Manuscript received on 20 February 2026 | Second Revised Manuscript received on 21 March 2026 | Manuscript Accepted on 15 April 2026 | Manuscript published on 30 April 2026.
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Jensen gaps in measure-theoretic frameworks and illustrated their applications to model selection and low-risk decision-making. Ajmal et al. [13] specifically highlighted the Caputo fractional operators in deriving inequalities for memory and nonlocal phenomena. Together, these studies illustrate a variety of fractional methods which apply to Jensen-type inequalities. They mostly apply fractional operators directly to both convex functions and expectation structures. To the best of our knowledge, no prior research offers an operator-level enhancement of the inequality that continuously parametrises the Jensen gap itself, which motivates this paper.

In this paper, a fractional-order version of Jensen's inequality is constructed using Riemann-Liouville operators. The method proposes an oriented fractional correction term that leads to a continuum of Jensen-type inequalities indexed by the fractional order, and the classical Jensen inequality is obtained as a limiting case. Its equivalent Caputo version is also given, and it is particularly well-suited to describing memory-dependent and nonlocal averaging processes, which play a fundamental role in economic and financial models with intertemporal dependencies.

We pay particular attention to the exponential integral function Ei(x), which is convex on (1, ∞) and has applications in applied mathematics and economic utility theory. Its role in growth, accumulation, and risk-seeking processes is illustrated by analyses of real-world exponential-type processes, such as depreciation, population and bacterial growth, and compound interest [14], as well as by numerically stable finite-difference schemes for European option pricing in the Black-Scholes model [15]. Moreover, fractional approaches and Caputo-type inequalities, as presented in [13], emphasise the role of memory and nonlocal interactions, and Ei(x) serves as a suitable test function for improved Jensen-type inequalities.

Fractional calculus, and specifically the Caputo fractional derivative, is a useful instrument for modelling non-local and memory properties in mathematical models [16]. Although the main result of our findings concerns Riemann-Liouville fractional integrals and their order-monotonicity, we include Caputo fractional integrals for modelling and completeness.

The contributions of this paper are summarized as follows:

- We derive a fractional-order version of Jensen's inequality for the exponential integral function by adding an oriented Riemann–Liouville correction term that produces a continuum of bounds labelled by a fractional parameter.
- We demonstrate that the classical Jensen gap is a limiting case of this family of bounds and present an equivalent Caputo formulation that emphasizes the operator- theoretic character of the result.

In this work, we prove the main inequality based on elementary properties of the ordering of the Riemann-Liouville fractional integral, which is a useful instrument when comparing different fractional orders. The Caputo version of the inequality (Eq. (3.5)) is mathematically equivalent (see Lemma 15). Still, we stress it here because it is more naturally compatible with the operator-based formulation and the way these concepts are applied in practice.

II. MATHEMATICAL PRELIMINARIES

A. Function Spaces and Notation

Definition 1 (Absolutely continuous classes AC^n). For an interval $[a,b], AC^1([a,b])$ denotes the set of absolutely continuous functions on $[a,b]$. For $n \geq 2, AC^n([a,b])$ consists of functions whose derivatives up to order $n-1$ are absolutely continuous and whose n- The derivative is in $L^1([a,b])$.

We write $\mathbf{1}_{\{\cdot\}}$ for indicator functions and sgn(·) for the sign function.

B. Foundations of Fractional Calculus

Fractional calculus extends classical differentiation and integration to non-integer orders, introducing memory and non-local effects into mathematical operations. The Riemann-Liouville approach emphasises order-monotonicity properties, while the Caputo formulation is often preferred for modelling physical and economic processes due to its compatibility with initial conditions.

Definition 2 (Riemann-Liouville Fractional Integral). Let $\alpha > 0$ and $f \in L^1([a,b])$. The left-sided Riemann-Liouville fractional integral of order α is

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a \quad \dots \quad (1)$$

Definition 3 (Caputo Fractional Derivative (left-sided)). Let $\alpha > 0$ and $n = \lceil \alpha \rceil \in \mathbb{N}$. Suppose $f \in AC^n([a,b])$. The left-sided Caputo fractional derivative of order α is

$$\left({}^{C}D_{a^+}^{\alpha} f\right)(x) = I_{a^+}^{n-\alpha}\left[\frac{d^n}{dx^n} f\right](x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt \quad \dots \quad (2)$$

Definition 4 (Caputo Fractional Derivative (right-sided)). For $0 < \alpha < 1$ and $f \in AC^1([a,b])$, the right-sided Caputo derivative with base $c \in [a,b]$ is

Retrieval Number:100.1/ijam.A123406010426
DOI: 10.54105/ijam.A1234.06010426
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

$$({}^{C}D_{c^-}^{\alpha} f)(x) := -\frac{1}{\Gamma(1-\alpha)} \int_x^c (t-x)^{-\alpha} f'(t) dt, x < c \quad \dots \quad (3)$$

Convention used later: when applying right-sided Caputo operators in 3, we take the base $c = \mu$ (the mean of X).

For example, applying the Caputo derivative to a power function $f(x) = x^p$ yields ${}^{C}D_{a^+}^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}$ for $x > a$, illustrating how fractional operators interpolate between integer-order derivatives.

With $\rho = 1 - \alpha \in (0,1)$, the RL kernel $(x-t)^{\rho-1}/\Gamma(\rho)$ places a higher weight on the values of t close to x. As $\alpha \uparrow 1$ (i.e., $\rho \downarrow 0$), this kernel concentrates near $t = x$, corresponding to a shorter effective memory. We therefore view α As a mathematical proxy for memory, it should be interpreted as an operational weighting of the past rather than a literal psychological mechanism.

C. The Exponential Integral Function and Its Properties

Definition 5 (Exponential Integral). For real, non-zero x,

$$\mathrm{Ei}(x) = p.v. \int_{-\infty}^{x} \frac{e^t}{t} dt \quad \dots \quad (4)$$

Lemma 6 (Properties of the Exponential Integral). For $x \neq 0$

: (i) $\frac{d}{dx} \mathrm{Ei}(x) = \frac{e^x}{x}$.

(ii) $\mathrm{Ei}(x)$ is strictly convex on $(1,\infty)$ and concave on $(0,1)$; at $x = 1$ we have $\mathrm{Ei}''(1) = 0$.

Proof. (i) is standard. For (ii),

$$\mathrm{Ei}''(x) = \frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{e^x(x-1)}{x^2}$$

which is < 0 on $(0,1)$, $= 0$ at 1, and > 0 on $(1,\infty)$.

Henceforth, we apply Jensen's inequality for Ei only on intervals contained in $[1,\infty)$ to ensure convexity. The Caputo operator is retained for modelling purposes, while the main inequality relies on RL integrals and their order-monotonicity.

D. A Positivity Lemma

Lemma 7. Let $\alpha \in (0,1)$, and let $f \in C^1([a,b])$ be non-decreasing. Then $\left({}^{C}D_{a^+}^{\alpha} f\right)(x) \geq 0$ for $x \in (a,b]$.

Proof. Immediate from the definition with $n = 1$ and $f'(t) \geq 0$.

E. Interchanging Expectation and Fractional Operators

Lemma 8 (Tonelli/Fubini conditions for oriented RL integrals). Let X have bounded support $[a,b] \subset [1,\infty)$. If $f \in L^1([a,b])$ is nonnegative and $\rho \in (0,1]$, then the mappings

$$x \mapsto \left(I_{\mu+}^{\rho} f\right)(x) \text{ on } (\mu,b], x \mapsto \left(I_{\mu-}^{\rho} f\right)(x) \text{ on } [a,\mu)$$

are measurable and integrable, and one may interchange $\mathbb{E}$ with the *RL* integrals defining $\left(I_{\mu\pm}^{\rho} f\right)(X)$.

Proof. On bounded support, kernels $(x-t)^{\rho-1}\mathbf{1}_{\{t \leq x\}}$ and $(t-x)^{\rho-1}\mathbf{1}_{\{t \geq x\}}$ are integrable against $f(t)dt$; Tonelli's theorem applies, yielding measurability and justified interchange.

F. Assumptions and Notation

Throughout, let X be a real-valued random variable with $\mathrm{supp}(X) \subset [1,\infty)$ and mean $\mu = \mathbb{E}[X]$. We write $f(t) = e^t/t$. When invoking limits as $\alpha \uparrow 1$ (i.e., $\rho \downarrow 0$), we additionally assume $\mathbb{E}[e^X/X] < \infty$ to justify dominated convergence. We denote by $I_{\mu\pm}^{\rho}$ the oriented Riemann-Liouville integrals of order $\rho = 1 - \alpha \in (0,1)$. Indicators $\mathbf{1}_{\{\cdot\}}$ and $\mathrm{sgn}(\cdot)$ follow standard usage.

Remark 9 (Extending below 1 via shifting). If $\mathrm{supp}(X) \not\subset [1,\infty)$, choose $c \in \mathbb{R}$ with $X + c \geq 1$ a.s. and set $Z := X + c, \mu_Z = \mathbb{E}[Z] = \mu + c$. Applying the main result to $g(z) = \mathrm{Ei}(z)$ and Z yields $\mathrm{Ei}(\mu_Z) \leq \mathbb{E}[\mathrm{Ei}(Z)] - M_Z(\alpha)$, with $M_Z(\alpha)$ defined as in (3.4) but centred at μ_Z. Translating back gives a bound for $g_c(x) := \mathrm{Ei}(x+c)$ on the original X, preserving convexity on the shifted domain.

III. MAIN RESULTS

Convention. Throughout this section, we write.

$$\mu := \mathbb{E}[X]$$

for the mean of the random variable X introduced below.

A. Mathematical Setup and Auxiliary Results

We begin by fixing the probabilistic framework.

Definition 10 (Probability Space and Random Variable). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $X: \Omega \to \mathbb{R}$ be an integrable random variable satisfying

$$\mathrm{supp}(X) \subset [a,b] \subset [1,\infty)$$

We denote its mean by

$$\mu = \mathbb{E}[X].$$

The following lemma provides a classical integral representation of the Jensen gap. It will serve as the structural basis for all subsequent refinements.

Lemma 11 (Integral Representation of the Jensen Gap). Let $g \in C^2([1,\infty))$. Then

$$\mathcal{J}(g) := \mathbb{E}[g(X)] - g(\mu) = \mathbb{E}\left[\int_\mu^X (X-s) g''(s) ds\right]$$

where the integral is interpreted in the oriented sense when $X < \mu$.

Proof. Since $g \in C^2([1,\infty))$, the second-order Taylor formula with integral remainder gives, for every x in the support of X,

$$g(x) = g(\mu) + g'(\mu)(x-\mu) + \int_\mu^x (x-s) g''(s) ds$$

Substituting $x = X$ yields

$$g(X) = g(\mu) + g'(\mu)(X-\mu) + \int_\mu^X (X-s) g''(s) ds$$

Taking expectations and using $\mathbb{E}[X-\mu] = 0$, we obtain

$$\mathbb{E}[g(X)] - g(\mu) = \mathbb{E}\left[\int_\mu^X (X-s) g''(s) ds\right]$$

The oriented interpretation of the integral ensures validity for all realizations of X.

We now specialize to the exponential integral.

Proposition 12 (Convexity of the

Exponential Integral on $[1,\infty)$). The exponential integral function $\mathrm{Ei}(x)$ is convex on $[1,\infty)$.

Proof. For $x>0$, the standard derivatives of the exponential integral are

$$\mathrm{Ei}'(x)=\frac{e^x}{x},\mathrm{Ei}''(x)=\frac{e^x(x-1)}{x^2}$$

For all $x\geq 1$, we have $x-1\geq 0$ and $e^x>0$, hence

$$\mathrm{Ei}''(x)\geq 0$$

Therefore, Ei is convex on $[1,\infty)$.

We may now compute the Jensen gap explicitly.

Corollary 13 (Jensen Gap for the Exponential Integral). Let X satisfy Definition 10. Then

$$\mathcal{J}(\mathrm{Ei})=\mathbb{E}[\mathrm{Ei}(X)]-\mathrm{Ei}(\mu)=\mathbb{E}\left[\int_{\mu}^{X}(X-s)\frac{e^s(s-1)}{s^2}ds\right] \quad \dots \quad (5)$$

and in particular,

$$\mathcal{J}(\mathrm{Ei})\geq 0$$

Proof. Applying Lemma 11 with $g=\mathrm{Ei}$ and using

$$g''(s)=\frac{e^s(s-1)}{s^2} \text{ Gives } \quad \dots \ (6).$$

Since Ei is convex on $[1,\infty)$ By Proposition 12, Jensen's inequality implies

$$\mathbb{E}[\mathrm{Ei}(X)]\geq \mathrm{Ei}(\mu),$$

and therefore $\mathcal{J}(\mathrm{Ei})\geq 0$.

B. Oriented RL Fractional Integrals and Order-Monotonicity

For $\rho>0$ and base $c\in\mathbb{R}$, define the oriented Riemann-Liouville fractional integrals

$$\left(I_{c+}^{\rho}h\right)(x):=\frac{1}{\Gamma(\rho)}\int_c^x(x-t)^{\rho-1}h(t)dt, x>c$$

$$\left(I_{c-}^{\rho}h\right)(x):=\frac{1}{\Gamma(\rho)}\int_x^c(t-x)^{\rho-1}h(t)dt, x<c$$

In particular,

$$I_{c+}^{1}h(x)=\int_c^x h(t)dt, I_{c-}^{1}h(x)=\int_x^c h(t)dt$$

It is useful to normalize the RL operator in order to reveal its probabilistic structure. Define the normalized oriented operator.

$$\mathcal{J}_{c+}^{\rho}h(x):=\frac{\Gamma(\rho+1)}{(x-c)^{\rho}}\left(I_{c+}^{\rho}h\right)(x), x>c$$

and analogously

$$\mathcal{J}_{c-}^{\rho}h(x):=\frac{\Gamma(\rho+1)}{(c-x)^{\rho}}\left(I_{c-}^{\rho}h\right)(x), x<c$$

Using $\Gamma(\rho+1)=\rho\Gamma(\rho)$ and the change of variables $t=x-(x-c)u, u\in[0,1]$,

We obtain the representation

$$\mathcal{J}_{c+}^{\rho}h(x)=\rho\int_0^1 u^{\rho-1}h(x-(x-c)u)du$$

$$=\mathbb{E}\left[h\left(x-(x-c)U_{\rho}\right)\right]$$

where U_{ρ} has the Beta$(\rho,1)$ density $\rho u^{\rho-1}$ on $(0,1)$.

Lemma 14 (Order-monotonicity of the normalized operator). Let $0<\rho_1\leq\rho_2\leq 1$.

1. If h is nondecreasing on $[c,x]$ and $x>c$, then

$$\mathcal{J}_{c+}^{\rho_1}h(x)\geq\mathcal{J}_{c+}^{\rho_2}h(x)$$

2. If h is nonincreasing on $[c,x]$ and $x>c$, then

$$\mathcal{J}_{c+}^{\rho_1}h(x)\leq\mathcal{J}_{c+}^{\rho_2}h(x)$$

3. Analogous statements hold for $\mathcal{J}_{c-}^{\rho}$ on the interval $[x,c]$.

Proof. The representation

$$\mathcal{J}_{c+}^{\rho}h(x)=\mathbb{E}\left[h\left(x-(x-c)U_{\rho}\right)\right]$$

shows that $U_{\rho_1}\leq_{\mathrm{st}}U_{\rho_2}$ for $0<\rho_1\leq\rho_2\leq 1$ (stochastic dominance of Beta distributions). If h is nondecreasing, then $u\mapsto h(x-(x-c)u)$ is nonincreasing. For nonincreasing functions, stochastic dominance reverses the expectation inequality, giving.

$$\mathbb{E}\left[h\left(x-(x-c)U_{\rho_1}\right)\right]\geq\mathbb{E}\left[h\left(x-(x-c)U_{\rho_2}\right)\right].$$

This proves part (1). Part (2) follows the same argument with the monotonicity of h reversed. The proof for $\mathcal{J}_{c-}^{\rho}$ is analogous.

C. Caputo-RL Identities on Both Sides of μ

The standard connection between Caputo derivatives and RL fractional integrals is as follows.

Lemma 15 (Caputo-RL identities on both sides of μ). Let $g\in C^1([1,\infty))$ and set $f=g'$. For $0<\alpha<1$ and $\rho=1-\alpha$:

$$\begin{aligned}\left({}^{C}D_{\mu+}^{\alpha}g\right)(x)&=\left(I_{\mu+}^{\rho}f\right)(x), & x>\mu,\\ \left({}^{C}D_{\mu-}^{\alpha}g\right)(x)&=-\left(I_{\mu-}^{\rho}f\right)(x), & x<\mu.\end{aligned} \quad \dots \quad (7)$$

Proof. For the left side, the Caputo definition with $n=1$ gives

$$\left({}^{C}D_{\mu+}^{\alpha}g\right)(x)=I_{\mu+}^{1-\alpha}g'(x)=\left(I_{\mu+}^{\rho}f\right)(x).$$

For the right side, a corresponding definition yield

$$\left({}^{C}D_{\mu-}^{\alpha}g\right)(x)=-I_{\mu-}^{1-\alpha}g'(x)=-\left(I_{\mu-}^{\rho}f\right)(x).$$

Lemma 16 (Limit of RL integrals as $\rho\downarrow 0$). If f is continuous on $[a,b]$, then

$$\left(I_{c+}^{\rho}f\right)(x)\to f(x) \text{ for } x>c, \left(I_{c-}^{\rho}f\right)(x)\to f(x) \text{ for } x<c$$

as $\rho\downarrow 0$. If $f\in L^1([a,b])$, the convergence holds in $L^1([a,b])$.

Sketch of proof. These limits follow from standard properties of Riemann-Liouville integrals as convolution operators: as the order. $\rho\to 0$, the kernel $(x-t)^{\rho-1}/\Gamma(\rho)$ approaches a Dirac mass at $t=x$, yielding pointwise convergence for continuous f and convergence in L^1 for integrable f; see [17] for detailed treatments.

D. The Main Theorem

Theorem 17 (Fractional Jensen Inequality for the Exponential Integral). Let $\alpha\in(0,1)$ and set $\rho:=1-\alpha\in(0,1)$. With $f(t)=e^t/t$, define the oriented fractional correction

$$\mathcal{M}(\alpha):=\mathbb{E}\left[\mathbf{1}_{\{X\geq\mu\}}\left(I_{\mu+}^{\rho}f\right)(X)-\mathbf{1}_{\{X<\mu\}}\left(I_{\mu-}^{\rho}f\right)(X)\right]. \quad \dots \quad (8)$$

Equivalently (Caputo form). By Lemma 15,

$$\mathcal{M}(\alpha)=\mathbb{E}\left[\mathbf{1}_{\{X\geq\mu\}}\left({}^{C}D_{\mu+}^{\alpha}\mathrm{Ei}\right)(X)+\mathbf{1}_{\{X<\mu\}}\left({}^{C}D_{\mu-}^{\alpha}\mathrm{Ei}\right)(X)\right]\dots \ (9)$$

Then, for X as in Definition 10,

$$\mathrm{Ei}(\mu)\leq\mathbb{E}[\mathrm{Ei}(X)]-\mathcal{M}(\alpha). \quad \dots \quad (10)$$

Furthermore, in situations where $\mathcal{M}(\alpha)\geq 0$ (which may occur under additional one-sided support assumptions),

$$\mathbb{E}[\mathrm{Ei}(X)]-\mathcal{M}(\alpha)\leq\mathbb{E}[\mathrm{Ei}(X)]. \quad \dots \quad (11)$$

Define the continuous extension. $\overline{\mathcal{M}}(0):=\lim_{\alpha\downarrow 0}\mathcal{M}(\alpha)$; then $\overline{\mathcal{M}}(0)=\mathcal{J}(\mathrm{Ei})$. If X is non-degenerate, the inequality in (3.6) is strict for every $\alpha\in(0,1)$.

Retrieval Number:100.1/ijam.A123406010426
DOI: 10.54105/ijam.A1234.06010426
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

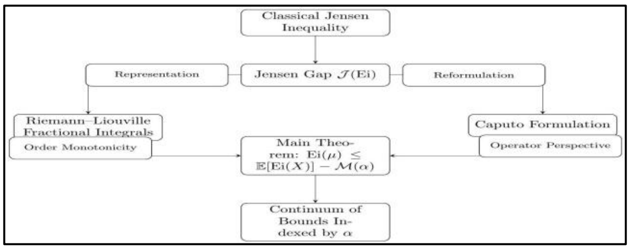
Advanced Journal of Mathematics
IJAM
Exploring Innovation
www.ijam.latticescipub.com

Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-6 Issue-1, April 2026

Remark 18 (On one-sided support conditions). The sufficient condition $\mathbb{P}(X \geq \mu) = 1$ forces X to be almost surely constant and is therefore not assumed in the main theorem. It is mentioned only to illustrate situations in which the correction term $\mathcal{M}(\alpha)$ is nonnegative and yields a two-sided bound. The inequality (3.6) holds without this restriction.

Corollary 19 (Convex increasing g). Let $g \in C^2([1,\infty))$ be convex and increasing, and let X satisfy the standing assumptions. With $f = g' \geq 0$ and $\rho = 1-\alpha \in (0,1)$, define $M_g(\alpha) := \mathbb{E}\left[\mathbf{1}_{\{X\geq\mu\}}\left(I_{\mu+}^{\rho} f\right)(X) - \mathbf{1}_{\{X<\mu\}}\left(I_{\mu-}^{\rho} f\right)(X)\right]$. Then $g(\mu) \leq \mathbb{E}[g(X)] - M_g(\alpha)$, and $M_g(0) = \mathcal{J}(g)$. If $P(X \geq \mu) = 1$, then $M_g(\alpha)$ is non-increasing in α and $\lim_{\alpha\uparrow 1} M_g(\alpha) = \mathbb{E}[g'(X)]$ by Lemma 16.

Proof roadmap. Figure 1 summarizes the argument: the classical Jensen gap representation feeds into oriented Riemann-Liouville integrals and an equivalent Caputo reformulation, and order monotonicity yields Theorem 17 and the continuum of bounds.



[Fig.1: Conceptual Flow from Classical Jensen Inequality to our Fractional Refinement. The Framework Provides a Continuum of Bounds Parameterized by the Memory Parameter α, Connecting Riemann-Liouville Integrals (Order Monotonicity) with Caputo Formulation (Operator Perspective)]

Proof. Justification of interchanges. By Lemma 8, we may interchange expectation with the oriented RL integrals in (3.4).
(Left inequality). By Lemma 11 with $g' = f$ and splitting on $\{X \gtrless \mu\}$,

$$\mathcal{J}(\mathrm{Ei}) = \mathbb{E}\left[\mathbf{1}_{\{X\geq\mu\}}\left(I_{\mu+}^{1} f\right)(X) - \mathbf{1}_{\{X<\mu\}}\left(I_{\mu-}^{1} f\right)(X)\right]$$

Since $0 < \rho < 1$ and $f \geq 0$, Lemma 14 yields $\left(I_{\mu\pm}^{\rho} f\right) \leq \left(I_{\mu\pm}^{1} f\right)$ on their domains. Taking expectations with the same signs gives $\mathcal{M}(\alpha) \leq \mathcal{J}(\mathrm{Ei})$. Rearranging,

$$\mathrm{Ei}(\mu) = \mathbb{E}[\mathrm{Ei}(X)] - \mathcal{J}(\mathrm{Ei}) \leq \mathbb{E}[\mathrm{Ei}(X)] - \mathcal{M}(\alpha).$$

(Conditional right inequality). If in addition $\mathcal{M}(\alpha) \geq 0$, then trivially $\mathbb{E}[\mathrm{Ei}(X)]$ $\mathcal{M}(\alpha) \leq \mathbb{E}[\mathrm{Ei}(X)]$.
(Monotonicity note). On $\{X \geq \mu\}$, the contribution to $\mathcal{M}(\alpha)$ is non-increasing in α (since I^{ρ} is order-nondecreasing in $\rho = 1-\alpha$). On $\{X < \mu\}$, the contribution is - $\left(I_{\mu-}^{\rho} f\right)$ which is non-decreasing in α. Global monotonicity of $\mathcal{M}$ in α is therefore not automatic without additional assumptions.
(Strictness). If X is non-degenerate, the order inequality used above is strict on a set of positive probability, so $\mathcal{M}(\alpha) < \mathcal{J}(\mathrm{Ei})$ and (3.6) is strict.

The implication is crucial for economic applications because it provides a mathematical model for measuring the impact of memory effects on risk estimation. The variable α is used as the memory variable, where $\alpha \to 0$ represents the classical situation where there is no memory, and $\alpha \to 1$ indicates path dependence. This continuum enables the modelling of different economic behaviours in which experiences influence decisions.

To demonstrate the characteristics of monotonicity, we analyze a simple situation where X has only two possible outcomes. Figure 2 illustrates the behaviour of $\mathcal{M}(\alpha)$ as a function of α for such a distribution. The curve shows the typical decreasing behavior of $\mathcal{M}(\alpha)$ with increasing α, with contributions from above and below the mean exhibiting complementary monotonicity properties.

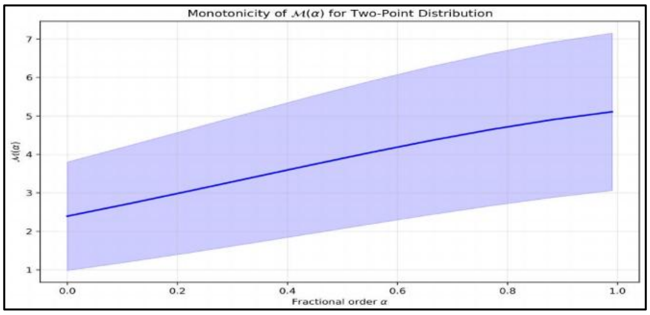


Fig.2: A Simple Two-Point Distribution Illustration of $\mathcal{M}(\alpha)$ Monotonicity]

Remark 20 (RL vs Caputo forms). The *RL* The form is convenient for order comparisons via Lemma 14. The Caputo form (3.5) emphasises the operator-theoretic view and aligns with the title. Both are exactly equivalent by Lemma 15.

Remark 21 (Limits of $\mathcal{M}(\alpha)$). As $\alpha \to 0^{+}, \rho \to 1$ and $\overline{\mathcal{M}}(0) = \mathcal{J}(\mathrm{Ei})$. As $\alpha \to 1^{-}$, $\rho \downarrow 0$ and $I_{\mu\pm}^{\rho} f(x) \to f(x)$, so under $\mathbb{E}[e^{X}/X] < \infty$,

$$\lim_{\alpha\to 1^{-}} \mathcal{M}(\alpha) = \mathbb{E}[f(X)\mathrm{sgn}(X-\mu)],$$

which is generally nonzero. If $\mathbb{P}(X \geq \mu) = 1$, the limit equals $\mathbb{E}[f(X)]$.

IV. NUMERICAL ILLUSTRATION

This section provides a numerical validation of the theoretical results in Section 3. First, we use a simple abstract setup to examine core properties of the fractional correction. We then consider a more economically realistic scenario.

Unless stated otherwise, we set the random seed to 2025 for all Monte Carlo experiments. RL integrals with weak endpoint singularities (for α near 1) are evaluated using productintegration on a graded mesh toward the endpoint. This method uses a transformation $t = x + (c-x)u^{\beta}$ (for left integrals) or $t = x + (c-x)\left(1-u^{\beta}\right)$ (for right integrals) with grading exponent $\beta \in [1.5,2]$, which clusters quadrature points near the singular endpoint to improve accuracy. We used at least $N_q = 200$ panels, increasing until the change between successive refinements was $< 10^{-4}$. Monte Carlo standard errors (SE) are reported as $s/\sqrt{N}$ and 95% confidence intervals (CI) as a point $\pm 1.96 s/\sqrt{N}$, where s is the sample standard deviation.

A. Verification of Essential Mathematical Properties

Figure 3 presents numerical results for $\mathcal{M}(\alpha)$ under a uniform distribution $X \sim U(1.1,4.0)$, chosen so that $X > 1$ ensures the convexity of $U(x) = \text{Ei}(x)$ on the support.

A Monte Carlo simulation with $N = 500$ Samples (seed 2025) yielded a mean $\mu \approx 2.5458$. The classical Jensen gap was estimated as

$$\mathcal{J}(\text{Ei}) = \mathbb{E}[\text{Ei}(X)] - \text{Ei}(\mu) \approx 1.2116$$

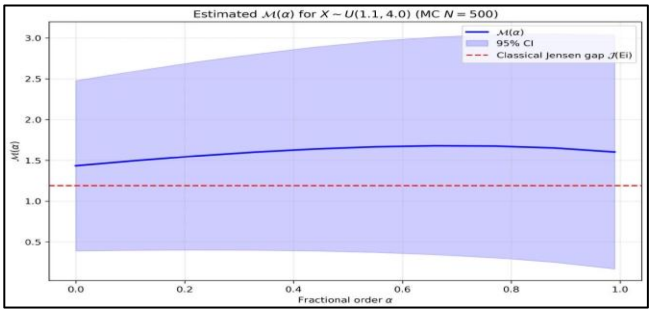
This equals the continuous extension $\overline{\mathcal{M}}(0)$ and serves as the starting point for the fractional correction.

In our examples $\mathcal{M}(\alpha)$ decreases with α, but this trend is empirical; beyond the one-sided case $P(X \geq \mu) = 1$ (where non-increase holds), Global monotonicity need not hold in general.

The computed values of $\mathcal{M}(\alpha)$ for $\alpha \in [0,1)$ are plotted in Figure 3, and selected values are tabulated in Table 1. The curves in this instance empirically exhibit a strictly decreasing trend in α (equivalently, nondecreasing in ρ) on both regions $\{X \geq \mu\}$ and $\{X < \mu\}$, leading to an overall decrease $\mathcal{M}(\alpha)$ for this distribution. This observation aligns with the theoretical regional order-monotonicity; global monotonicity is not claimed in full generality without additional assumptions. Error bands (shown in light blue) correspond to $\pm 1.96\text{SE}$ computed from the Monte Carlo sample. The curve decreases with α, consistent with regional order-monotonicity. The continuum of bounds provides increasingly tighter estimates as α increases.

Table 1: Fractional correction $\mathcal{M}(\alpha)$ for $X \sim U(1.1,4.0)$ (MC$N = 500$, seed 2025). Parentheses show Monte Carlo SE $= s/\sqrt{N}$; 95% CIs are point $\pm 1.96 s/\sqrt{N}$.

Fractional Order α	**$\mathcal{M}(\alpha)$ (Estimate)**	**SE**
0.00	1.2116	(0.0321)
0.25	1.1462	(0.0304)
0.50	1.0849	(0.0287)
0.75	1.0277	(0.0272)
0.90	0.9955	(0.0263)
0.99	0.9772	(0.0259)



[Fig.3: Estimated $\mathcal{M}(\alpha)$ for $X \sim U(1.1, 4.0)$ with MC$N = 500$ (seed 2025)]

B. Economic Interpretation in a Realistic Scenario

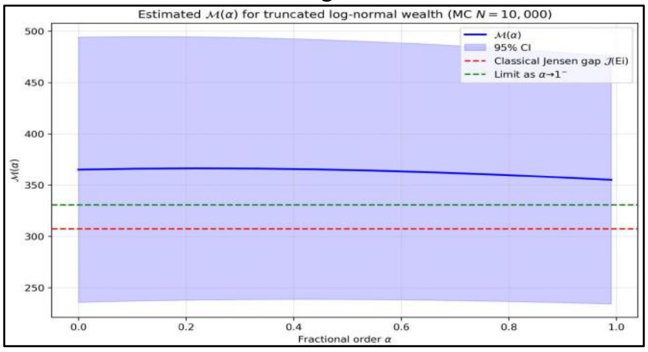
Considering a right-skewed wealth distribution, we let Y follow a truncated log-normal distribution with log $(Y) \sim \mathcal{N}(3,1)$, truncated to $[1,10]$. AWe observes that a sample of $N = 10{,}000$ realizations (seed 2025) yields $\hat{\mu} \approx 3.4$. The classical Jensen gap is substantially larger:

$$\mathcal{J}(\text{Ei}) = \mathbb{E}[\text{Ei}(Y)] - \text{Ei}(\hat{\mu}) \approx 288.09.$$

The parameter α can be interpreted as a memory parameter in economic decision-making. For example, in recursive utility models, α could represent the degree to which past consumption experiences influence current utility assessments. With α is close to 0, decision-making is based on long-term averages (classical expected utility), and when α approaches 1, current experiences have greater influence, leading to more path-dependent utility evaluations. This might capture scenarios in which investors' risk preferences are driven by recent market outcomes, or in which consumers' satisfaction depends on current consumption experiences relative to past averages.

Our model, under risk-seeking behavior (conveyed by the convexity of Ei on $[1, \infty)$), illustrates how memory effects can either strengthen or weaken risk-seeking behaviour. Decision-makers with high memory effects (α close to 1) might show more volatile risk preferences driven by recent outcomes, whereas those with low memory effects (α close to 0) tend to rely on risk assessments based on averages.

Figure 4 plots $\mathcal{M}(\alpha)$, which gives a sense of the continuum of bounds depending on α. The parameter α indexes memory: $\alpha = 0$ gives the classical bound $\mathcal{J}(\text{Ei})$; for intermediate α, memory fades; and $\alpha \to 1^-$approaches the strongest path-dependent bound, with a limit equal to an estimate of $\mathbb{E}[(e^Y/Y)\text{sgn}(Y - \hat{\mu})]$ for this model. Error bands (in light blue) represent $\pm 1.96\text{SE}$ around the Monte Carlo estimate. The limit as $\alpha \to 1^-$equals a Monte Carlo estimate of $\mathbb{E}[(e^Y/Y)\text{sgn}(Y - \hat{\mu})]$. The continuum of bounds shows how memory effects affect risk assessment in economic decision-making.



[Fig.4: Estimated $\mathcal{M}(\alpha)$ for Truncated log-normal Wealth (MC$N = 10{,}000$, seed 2025)]

V. CONCLUSION

In this work, we have derived a fractional-order version of Jensen's inequality for the exponential integral function, which takes into account the memory effects via the parameter α. Our main result states that for X concentrated in $[1, \infty)$ and $\alpha \in (0,1)$,

$$\text{Ei}(\mathbb{E}[X]) \leq \mathbb{E}[\text{Ei}(X)] - \mathcal{M}(\alpha),$$

where $\mathcal{M}(\alpha)$ is expressed via oriented Riemann-Liouville fractional integrals (or equivalently, Caputo fractional derivatives). This gives a continuum of bounds labeled by α, with the standard Jensen inequality being the $\alpha \to 0^+$limit and a path-dependent bound appearing at the $\alpha \to 1^-$limit.

The main novelty of this research consists of this continuum of bounds, which connects classical convex analysis with fractional calculus and provides a

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mathematical tool for describing memory effects in economic decision-making. For $\mathcal{M}(\alpha) \geq 0$, we have the two-sided inequality $\mathrm{Ei}(\mu) \leq \mathbb{E}[\mathrm{Ei}(X)] - \mathcal{M}(\alpha) \leq \mathbb{E}[\mathrm{Ei}(X)]$, which gives both lower and upper bounds on the expected utility.

In contrast to vanishing-limit results, $\mathcal{M}(\alpha)$ does not vanish as $\alpha \to 1^-$; instead, it tends to $\mathbb{E}[(e^X/X)\mathrm{sgn}(X-\mu)]$, which describes path-dependent features of the distribution. This storyline proposes a possible use of fractional operators to describe non-local averaging in convex functional evaluations, without referring to a particular behavioural or econometric theory.

VI. LIMITATIONS OF THE STUDY

The overriding limitation of this study is that the fractional refinement is based on the convexity of the exponential integral function on the support of the underlying random variable, which restricts the possible distributions to those with support on an interval where the exponential integral function is convex. Hence, both the mathematical findings and the illustrative examples are linked to this property of the exponential integral function.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted objectively and without external influence.
- **Ethical Approval and Consent to Participate:** The content of this article does not necessitate ethical approval or consent to participate with supporting documentation.
- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed solely by the author.

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