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Study and Consequences of the Ⓢ Function on the Riemann Hypothesis

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***Abstract** : I will study the Sghiar's function* $\circledS: (X,z) \mapsto \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{X}{p^z}}$, *$\mathcal{P}$ the set of prime numbers. Which is an extension of the Riemann zeta function. The classical form of the Riemann zeta function and its Euler product are well known in analytic number theory [1], and I will show that :* $\zeta(s) = 0$ *and* $Re(s) > \frac{1}{2} \Rightarrow \circledS = 0$. *We deduce the proof of the Riemann Hypothesis.*

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***Keywords:** Prime Number, Holomorphic Function, the Riemann Hypothesis.*

I. INTRODUCTION

If $|X| \leq 1$ and $\mathrm{Re}(z) > 0$ it can be seen that $\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z} = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$, where the trace $T(n) = \sum_i \alpha_i$ if $n = \prod_i n_i^{\alpha_i}$.

For the case where $X = 1$ and $\mathrm{Re}(z) > 0$, it was already known, by using the Dirichlet eta function, that $\prod_{p\in\mathcal{P}} \frac{1}{1-1/p^z} = \zeta(z)$ admits an analytic continuation to the complex plane, where it is holomorphic except for a simple pole at $z = 1$ [1].

However, in the general case, and even just for the case where $|X| < 1$ and $\mathrm{Re}(z) > 0$, the situation becomes more complicated, and it was necessary to prove holomorphy by a different method .

As we will see, the product $\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$, where it is defined, retains the same convergence and holomorphic properties as the product $\prod_{p\in\mathcal{P}} \frac{1}{1-1/p^z}$ This can be interpreted by noting that multiplying $\frac{1}{p^z}$ by X does not disrupt the convergence, and the holomorphic nature with respect to X and z is preserved.

The holomorphy of the function $\circledS: (X,z) \mapsto \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{X}{p^z}}$ with $\mathcal{P}$ being the set of prime numbers, will be investigated in this work. This analysis relies on classical properties of holomorphic functions and on known properties of the Riemann zeta function [1].

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I recall that related analytical approaches to the Riemann Hypothesis have been extensively developed in the modern literature on analytic number theory, particularly in recent works such as Conrey [1], Soundararajan [2], and Harper [2]. Furthermore, spectral and quantum-inspired perspectives related to the distribution of zeros have been investigated in recent studies on L-functions and automorphic forms (Radziwiłł & Soundararajan [4]; Sarnak [5]).

II. STATEMENT OF THE RESULT

Theorem : $\zeta(s) = 0$ and $\mathrm{Re}(s) > \frac{1}{2} \Rightarrow \circledS = 0$

Definition : Let $\mathcal{S} = \{(X,z) \in \mathbb{C}\times\mathbb{C}, 1 \neq |X/p^z|, \forall p \in \mathcal{P}\}$ with $\mathcal{P}$ being the set of prime numbers.

Lemma 1 :

In $\mathcal{S}$, $|\circledS(X,z)| < \infty$, where $\circledS: (X,z) \mapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$, $\mathcal{P}$ the set of prime numbers.

Proof of the Result :

For z and X complex numbers with $R(z) > 0$, we can observe the convergence of $\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ in $\mathcal{S}$ like this :

Let ϵ be a positive real number such that $\epsilon < R(z)$ and $R(z) - \epsilon \neq 1$.

Let p_0 be the smallest prime number such that $|X| < p_0^{\epsilon}$.

For all $p \in \mathcal{P}\ with\ p > p_0$, we have: $\left|\frac{1}{1-X/p^z}\right| \leq \left|\frac{1}{1-1/p^{R(z)-\epsilon}}\right|$.

It follows that :

$$\left|\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}\right| = \left|\prod_{p\in\mathcal{P},p\leq p_0} \frac{1}{1-X/p^z}\right| \cdot \left|\prod_{p\in\mathcal{P},p> p_0} \frac{1}{1-X/p^z}\right| \leq \left|\prod_{p\in\mathcal{P},p\leq p_0} \frac{1}{1-X/p^z}\right| \cdot \left|\prod_{p\in\mathcal{P},p> p_0} \frac{1}{1-1/p^{R(z)-\epsilon}}\right|.$$

Thus:

$$\left|\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}\right| \leq \left|\prod_{p\in\mathcal{P},p\leq p_0} \frac{1}{1-X/p^z}\right| . \frac{1}{\left|\prod_{p\in\mathcal{P},p\leq p_0} \frac{1}{1-1/p^{R(z)-\epsilon}}\right|} . \left|\zeta(R(z)-\epsilon)\right| < \infty$$

Lemma 2 :

For (X,z) in $\mathcal{S}$ with $R(z) > 0$, the function $\circledS: (X,z) \mapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ is holomorphic with respect to X and z .

The proof :

Let $\prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z} = \prod_{p\in\mathcal{P}} \left(1 + f_p(z)\right)$.

We have to show that the series $\sum_p \mid f_p(z) \mid$ converges normally on any compact set around X or z , with $f_p(z) = \frac{1}{(p^z/X)-1}$, which is equivalent to X/p^z for sufficiently large p.

From **lemma 1**, it has been shown that locally, around X or z, depending on the case,

$$\left|\prod_{p\in\mathcal{P}} \frac{1}{1 - X/p^z}\right| < \infty.$$

Thus, in **particular,** we have: $\lim_{q\to+\infty} \left|\prod_{p\in\mathcal{P},p>q} \frac{1}{1-|X|/p^{\mathrm{R}(z)}}\right| = 1.$

By using the logarithm and the equivalence between $\log\left(1 - |X|/p^{\mathrm{R}(z)}\right) and (|X|/p^{\mathrm{R}(z)} = |X/p^z|$, it follows that: $\lim_{q\to+\infty} \sum_{p\in\mathcal{P},p>q} |X/p^z| = 0.$

Moreover, using the equivalence between $f_p(z) = \frac{1}{(p^z/X)-1}$ and X/p^z for sufficiently large p, we deduce: $\lim_{q\to+\infty} \sum_{p\in\mathcal{P},p>q} \left|f_p(z)\right| = 0.$

Proof of the Theorem:

The link between the function ζ and the prime numbers had already been established by Leonhard Euler with the formula [1], valid for $Re(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1 - p^{-s}}$$
$$= \frac{1}{\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)}$$

where the infinite product is extended to the set $\mathcal{P}$ of prime numbers. This formula is sometimes called the Eulerian product.

Note : That mathematically and without worrying about convergence for the moment, even for $Re(s) > 0$, we can still see the equality above.

And since the Dirichlet eta function can be defined by $\eta(s) = (1 - 2^{1-s})\zeta(s)$ where : $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$

We have in particular : $\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$ for $0 < Re(z) < 1$,

As it is known that the non-trivial zeros of the Riemann zeta function are located in the critical strip where $0 < Re(s) < 1$, [1]

Let : $s = x + iy$, with $0 < Re(s) < 1$

$$\zeta(s)\zeta(\bar{s}) = \prod_{p\in\mathcal{P}} \frac{1}{1 - p^{-s}} \frac{1}{1 - p^{-\bar{s}}}$$
$$= \prod_{p\in\mathcal{P}} \frac{1}{\left(1 - e^{-x\ln(p)} cos\left(y\ln(p)\right)\right)^2 + \left(e^{-x\ln(p)} sin\left(y\ln(p)\right)\right)^2}$$

But : $\prod_{p\in\mathcal{P}} \frac{1}{\left(1-e^{-x\ln(p)}cos\left(y\ln(p)\right)\right)^2 + \left(e^{-x\ln(p)}sin\left(y\ln(p)\right)\right)^2} \geq \prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-x\ln(p)}\right)\right\}^2 + \left\{\left(e^{-x\ln(p)}\right)\right\}^2}$

If $\zeta(s) = 0$, $\prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-x\ln(p)}\right)\right\}^2 + \left\{\left(e^{-x\ln(p)}\right)\right\}^2} = 0$ and since the non-trivial zeros of ζ are symmetric with respect to the line $X = \frac{1}{2}$ because the zeta function satisfies the functional equation [1] :

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)$$

Then $x = \frac{1}{2} + \alpha$ and if $s' = \frac{1}{2} - \alpha + iy$, then $\zeta(s') = 0$

But the function $\frac{1}{\left\{\left(1+e^{-t\ln(p)}\right)\right\}^2 + \left\{\left(e^{-t\ln(p)}\right)\right\}^2}$ is increasing in [0,1], so $\prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-t\ln(p)}\right)\right\}^2 + \left\{\left(e^{-t\ln(p)}\right)\right\}^2} = 0 \quad \forall t \in [\frac{1}{2} - \alpha, \frac{1}{2} + \alpha]$.

As $\prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-z\ln(p)}\right)\right\}^2 + \left\{\left(e^{-z\ln(p)}\right)\right\}^2}$ is holomorphic : because :

$\prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-z\ln(p)}\right)\right\}^2 + \left\{\left(e^{-z\ln(p)}\right)\right\}^2} = \prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} \frac{1}{1-B/p^z}$

with $A = i - 1$ and $B = -i - 1$, and by **lemma 2**, both $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ and $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ are holomorphic in $\{z \in \mathbb{C} \setminus \{1\}, \mathrm{R}(z) \geq \frac{1}{2}\}$.

If $\alpha \neq 0$, then the holomorphic function $\prod_{p\in\mathcal{P}} \frac{1}{\left\{\left(1+e^{-z\ln(p)}\right)\right\}^2 + \left\{\left(e^{-z\ln(p)}\right)\right\}^2}$ will be null (because null on $]\frac{1}{2}, \frac{1}{2} + \alpha]$), and it follows that $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ or $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ is null in $\{z \in \mathbb{C} \setminus \{1\}, \mathrm{R}(z) \geq \frac{1}{2}\}$ (Because the zeros of a nonzero holomorphic function are separated [6]) . Let's show that this is impossible:

If $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} = \prod_{p\in\mathcal{P}} \left(1 + f_p(z)\right) = 0$ with $f_p(z) = \frac{1}{(p^z/A)-1}$ $\forall z \in \mathbb{C} \setminus \{1\}$, with $\mathrm{R}(z) \geq \frac{1}{2}$. So by the **Lemma 2** above, the application:

$\mathcal{S}: (X, z) \mapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ is holomorphic in the open quasi-disc $\mathcal{D} = \{X \in \mathbb{C}, 0 <\mid X \mid< \sqrt{2}\}$ with $z \in \mathbb{C}\setminus\{1\}, \mathrm{R}(z) \geq \frac{1}{2}$. (here z is fixed).

Let's extend the function $\mathcal{S}$ by setting :

For $z \in \mathbb{C}\setminus\{1\}, \mathrm{R}(z) \gneqq \frac{1}{2}$ and $\forall s \in \mathbb{R}, s \leq 0$, such as $\mathrm{R}(s + z) \geq 0$:

$\circledS(C/q^s, z) = \prod_{p\in\mathcal{P}} \frac{1}{1-C/(q^s p^z)}$. (where q is a number ≥ 2 , and C is such that $\mid C \mid= \sqrt{2}$) .

Note : We can see that for a small s we have $(C/q^s, z) \in \mathcal{S}$

In particular we have :

$\mathcal{S}(A/q^s, z) = \prod_{p\in\mathcal{P}} \frac{1}{1-A/(q^s p^z)}$ (where q is a prime number)

But for $z \in \mathbb{R} \setminus \{1\}, z \gneqq \frac{1}{2}$ we have :

$$\prod_{p\in\mathcal{P}} \left| \frac{1}{1 - A/(q^s p^z)} \right| \leq \prod_{p\in\mathcal{P}} \left| \frac{1}{1 - A/(p^z)} \right|$$

It follows that : $\mathcal{S}(A/q^s, z) = 0$, So : $\mathcal{S}(X, z) = 0, \forall X \in \mathcal{D}$

And consequently : $\mathcal{S} = 0$

Corollary 1 :

$\zeta(s) = 0$ *and* $\mathrm{R}(s) > 0 \Rightarrow \mathrm{R}(s) = \frac{1}{2}$

Proof:

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By contradiction, if this is not the case, due to the symmetry induced by the functional equation of the Riemann zeta function with respect to the critical line $\text{Re}(s) = \frac{1}{2}$

(See [1]), assuming that $\text{R}(s) > \frac{1}{2}$, From the **Theorem**, we would have $\mathscr{S} = 0$, and consequently $\zeta = 0$, which is absurd.

Connection between $\mathscr{S}(X,z)$, $\zeta(X,z)$, $\eta(X,z)$ and the zeros of $\zeta(X,z)$

If $|X| \leq 1$ and $\text{R}(z) > 0$, it can be seen that: $\mathscr{S}(X,z) = \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z} = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$

where the trace $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$.

Let us define: $\zeta(X,z) = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$, where the trace $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$.

Let us define: $\eta(X,z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^{T(n)}}{n^z}$, where the trace $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$.

Corollary 2 :

If $1 - 2^{1-z}X \neq 0$, then $\zeta(X,z)$ converges if and only if $\eta(X,z)$ converges, and we have: $(1 - 2^{1-z}X)\zeta(X,z) = \eta(X,z)$.

And $\zeta(X,z) = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$ where $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$, can be extended by : $\zeta(X,z) = \mathscr{S}(X,z)$ if $\text{R}(z) > 0$.

Proof:

$$
\begin{aligned}
(1 - 2^{1-z}X)\zeta(X,z) &= \zeta(X,z) - 2^{1-z}X\zeta(X,z) \\
&= \zeta(X,z) - 2^{1-z}\sum_{n=1}^{\infty} \frac{XX^{T(n)}}{n^z} \\
&= \zeta(X,z) - 2\sum_{n=1}^{\infty} \frac{X^{T(n)+1}}{\{2n\}^z} \\
&= \zeta(X,z) - 2\sum_{n=1}^{\infty} \frac{X^{T(2n)}}{\{2n\}^z} \\
&= \sum_{n=1}^{\infty} \frac{X^{T(2n+1)}}{\{2n+1\}^z} - \sum_{n=1}^{\infty} \frac{X^{T(2n)}}{\{2n\}^z} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^{T(n)}}{n^z} \\
&= \eta(X,z)
\end{aligned}
$$

Let us note that for $X = 1$, the convergence and holomorphy of $\eta(X,z)$ allowed the extension of the function ζ and demonstrated its holomorphy, which becomes less straightforward to use when $X \neq 1$. However, we managed to extend

$\zeta(X,z) = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$, where $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$, via: $\zeta(X,z) = \mathscr{S}(X,z)$ if $\text{R}(z) > 0$ since $\mathscr{S}(X,z)$ is holomorphic according to **Lemma 2**.

III. CONCLUSION AND EMERGENCE OF A NEW PROBLEM

Indeed, the introduction of $\mathscr{S}$ allowed me to solve the Riemann Hypothesis and to provide an extension of $\zeta(X,z) = \sum_{n=1}^{\infty} \frac{X^{T(n)}}{n^z}$, where $T(n) = \sum_i \alpha_i$ if $n = \prod n_i^{\alpha_i}$, via: $\zeta(X,z) = \mathscr{S}(X,z)$ if $\text{R}(z) > 0$. However, I have no idea where the zeros of $\zeta(X,z)$ are located when $X \neq 1$. A new problem has thus arisen !

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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- **Author's Contributions:** The authorship of this article is contributed solely.

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AUTHOR'S PROFILE

Mohamed Sghiar is an independent researcher and mathematician. He is a former professor at the Rectorate of the Dijon Academy and is currently a professor of mathematics and advanced (specialized) mathematics at the *École des Métiers* (France) (http://www.ecoledesmetiers.fr/).

Deeply passionate about both **mathematics and physics**, Mohamed Sghiar has devoted a significant part of his research to the **Riemann Hypothesis**, a central problem in number theory with profound connections to **quantum mechanics**. This interdisciplinary perspective explains why many of his mathematical proofs incorporate physical intuition and concepts drawn from mathematical physics.

Academic Background and Research Milestones

In **1990**, Mohamed Sghiar obtained a *Master's degree in Fundamental Mathematics* from **Hassan II University of Casablanca (Morocco)**. During this period, he acquired a strong foundation in **holomorphic functions**, a key tool that would later play an essential role in his investigations of the Riemann Hypothesis.

In **1993**, he earned a *DEA (Diplôme d'Études Approfondies) in Mathematics* from the **University of Dijon (France)**. He was particularly influenced by his professor of mathematical physics, **Moshe Flato**, whose encouragement fostered his strong interest in **mathematical physics**, especially **relativity** and **quantum mechanics**. This intellectual influence later led to the publication of two research articles in physics:

- *Quantum Gravity and the Two Forces Governing the Cosmos*, **IOSR Journal of Applied Physics**, 2017, Vol. 9, Issue 5, pp. 58–63.
- *The Mystery of Quantum Entanglement and Quantum Gravity Finally Elucidated*, **IOSR Journal of Applied Physics**, 2017, Vol. 9, Issue 5 (Version 4), pp. 24–27.

In **1994**, Mohamed Sghiar enrolled at **Université Paris-Sud (Orsay)** under the supervision of **Professor Jean-Guillaume Hagendorf**, working on problems in **logic and combinatorics**. Following Professor Hagendorf's illness and subsequent withdrawal from research, Mohamed Sghiar continued his work under the direction of **Professor Maurice Pouzet**, head of the laboratory. This collaboration resulted in several significant publications, including:

- *A Proof of Ulam's Conjecture*, **IOSR Journal of Mathematics**, 2016, Vol. 12, Issue 5, pp. 77–79.
- *(-1)-Reconstruction of Symmetric Graphs with at Least Three Vertices*, **IOSR Journal of Computer Engineering**, 2020, DOI: 10.9790/0661-2203031316.
- *(-3)-Reconstruction of Tournaments with at Least 14 Elements*, HAL preprint.
- *Measure and Action of i-Permutations on Finite and Infinite Multicolored Multigraphs*, arXiv preprint.

Independent Research Contributions

As an **independent researcher**, Mohamed Sghiar has continued to publish extensively in mathematics and theoretical computer science. His work addresses major open problems, including the **P = NP conjecture**, the **Birch and Swinnerton-Dyer conjecture**, and other fundamental conjectures at the interface of algebra, logic, and graph theory. Among his notable publications are:

- *A Proof of the Birch and Swinnerton-Dyer Conjecture*, **IOSR Journal of Mathematics**, 2018, Vol. 14, Issue 3, pp. 50–59.
- *Adjoint Action on Graphs and a Proof of the P = NP Conjecture*, **IOSR Journal of Computer Engineering**, Vol. 22, Issue 3, Series II, pp. 46–49.

Scientific Visibility Mohamed Sghiar's scientific works are available on major academic platforms, including **ResearchGate** and **HAL (Hyper Articles en Ligne)**, ensuring broad international visibility and accessibility of his research.