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Confirm Goldfeld Conjectured for Infinitely Many Elliptic Curves

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***Abstract:** Goldfeld conjectured: "a positive proportion of quadratic twists of an elliptic curve E/Q have an analytic rank of 1. In this work, we confirm this assertion for infinitely many elliptic curves.*

***Keywords.** Elliptic Curves, Goldfeld Conjectured, Analytic Rank, Optimal Elliptic Curve, Modular Curve and Binary Goldbach Problem.*

I. INTRODUCTION

Elliptic curves are fundamental objects in number theory and have critical applications in diverse fields, including cryptography, the study of Diophantine Equations, …

A general Weierstrass equation defines an Elliptic curve over Q:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\,;\, a_i \in \mathbb{Z}$$

It has the $j-invariant$ and $discrimiant$ Δ (See [1]), such that:

$$b_2 = a_1^2 + 4a_2\,, b_4 = 2a_4 + a_1a_3, b_6 = a_3^2 + 4a_6, b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$$

$$c_4 = b_2^2 - 24b_4, c_6 = -b_2^2 + 36b_2b_4 - 21b_6 \text{ and } \Delta = -c_4^3 - 27c_6^2$$

Let $E/\mathbb{Q}$ be an $elliptic\ curve$ and let $L(s,E) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be its $Hasse-weil\ L-function$ defined for $\Re(s) > \frac{3}{2}$. Let D be the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$, and let $\varkappa_D = \left(\frac{D}{.}\right)$ Denote usual $Kornecker\ Character$.

$Goldfeld\ conjectured$ that

$$\sum_{|D|<X} ord_{s=1}L(s,E_D) \sim \frac{1}{2}\sum_{|D|<X} 1$$

A $weaker\ Version$ of the conjectures is that for $r = 0\ or\ 1$,

$$\#\{|D| < X | ord_{s=1}L(s,E_D) = r\} \gg X$$

For $r = 0$, there is remarkable progress (see [3]). For $r = 1$, the results are much lower, $Vatsal\ and\ Byeon$ [3] confirmed this conjecture for only two $elliptic\ curves$. Then $Byeon, Joen\ and\ Kim$ [3] Provided results in this field that we rely on in our work.

Theorem 1. For a positive proportion of fundamental discriminants D, there are infinitely many $E/\mathbb{Q}$ such that

$$ord_{s=1}L(s,E_D) = 1$$

This theorem answers $problem$ 9.33 in [5].

II. ASSUMPTIONS

In this section, we introduce the objects we need later and collect some crucial facts about them.

Let $E/\mathbb{Q}$ be an $elliptic\ curve$ of conductor N and $X_0(N)$ the $modular\ curve$ of level N with $Jacobian\ J_0(N)$. The work of Darmon, Henri, and Rotger, Victor [4] shows that there is a $surjective\ morphism$ $\emptyset: X_0(N) \to E$ defined over $\mathbb{Q}$, which uniquely factors in $J_0(N)$ through a $homomorphism$ $\pi: J_0(N) \to E$. An $elliptic\ curve$ $E/\mathbb{Q}$ is said be $optimal$ if ker (π) is connected. There is a unique $optimal\ elliptic\ curve\ E$ in any $isogeny\ class$ of $elliptic\ curves$ defined over $\mathbb{Q}$ of conductor N.

Theorem 2.1. (see [2]) let $E/\mathbb{Q}$ be an $optimal\ elliptic\ curve$ of $square-free$ conductor N. Let F be the associated new form, and for $d|N$ let $w_d = \pm 1$ be such that $W_dF = w_dF$ where W_d is the $Atkin-Lehner\ involution$. Suppose that:

(i) $N = p.q$, where p, q There are different primes such that $w_p = -1, w_q = 1$ and $p \neq 3\,, q \equiv -1\ (mod\ 9)$

(ii) There is an $elliptic\ curve$ $E'/\mathbb{Q}$ Which is isogenous over $\mathbb{Q}$ to E and has a $\mathbb{Q}-rational$ $3-torsion$ Point.

Then $ord_{s=1}L(s,E_D) = 1$, for a positive proportion of fundamental discriminants D.

Let $G(x) \in \mathbb{Z}[x]$ Be a polynomial of degree k With a positive leading coefficient. $Perelli$ [3] and $Brudern, Kawada\ and\ Wooley$ [5] proved that almost all values of the polynomial $2G(m)$ Are of two primes.

Theorem 2.2. (See [2]) Let $G(x) \in \mathbb{Z}[x]$ Be a polynomial of degree k with a positive leading coefficient and $A\,, B$ be positive integers such that $(A,B) = 1$. Let $S_k(M,G)$ Denote the number of natural numbers m with $1 \leq m \leq M$ for which the equation

$$2G(m) = Ap_1 + Bp_2$$

Has no solution in primes p_1, p_2. Then there is an absolute constant $c > 0$ such that

$$S_k(M,G) \ll_G M^{1-c/k}$$

We use Theorem 2.2 to show that there are infinitely many $E/\mathbb{Q}$ Satisfying the conditions in Theorem 2.1.

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III. NEW RESULTS

Our principle is the following. Let p_1, p_2 Be two primes, both greater than 3 and $(p_1, p_2) = 1$. Let

$$E'/\mathbb{Q} : y^2 + 2p_1xy + p_2y = x^3$$

***Proposition* 3.1.** $(0,0) \in E'/\mathbb{Q}$ is $3 - torsion$ Point.

Proof. We easily see that $(0,0) \in E'/\mathbb{Q}$. The tangent $y = 0$ only intersects the curve at $(0,0)$, so $2(0,0) = \infty$. Then $3(0,0) = 2(0,0) + (0,0) = \infty + (0,0) = (0,0)$.

The discriminant $\Delta = p_2^3(p_1^2 - 27p_2)$, then $2, 3 \nmid \Delta$ and $c_4 = p_1^3(p_1^3 - 24p_2)$, We easily see that for every prime factor t of Δ, $E'/\mathbb{Q}$ has $multiplicative\ reductions$ at t. Thus, the conductor N of E' is $square - free$.

$(0,0)$ is a normal node at p_2, then $E'/\mathbb{Q}$ has $split\ multiplicative\ reduction$ at p_2, and $w_p = -1$.

It is a node; the slopes of the tangent lines at this node are $\frac{-p_1 \pm p_1\sqrt{-3}}{12}$: $\left(-\frac{p_1^2}{9}, -\frac{p_1^3}{27}\right)$

For $t \equiv 1\ (mod\ 3)$ (We find that $\sqrt{-3} \in F_t$), that implies $E'/\mathbb{Q}$ has a $split\ multiplicative\ reduction$ at t.

For $t \equiv -1\ (mod\ 3)$ (We find that $\sqrt{-3} \notin F_t$), that implies $E'/\mathbb{Q}$ has a $non - split\ multiplicative\ reduction$ at t.

We take $G(x) = (18x + 8x)^3/2$. By $Theorem$ 2.2, there are infinitely many m such that

$$2G(m) = (18m + 8)^3 = 27p + q$$

For some p, q are primes, we put $p_2 = p, p_1 = 9m + 4, m \in \mathbb{Z}^+$, then $\Delta = p^3.q$ and $N = p.q$ where $q = 9p_1^3 - 27p_2 \equiv -1\ (mod\ 9)$. So E' has $split\ multiplicative\ reduction$ at p and $non - split\ multiplicative\ reduction$ at q, then $w_p = -1, w_q = +1$. Thus, if we let $E/\mathbb{Q}$ be the $optimal\ elliptic\ curve$ of the $isogeny\ class$ of $E'/\mathbb{Q}$, then E satisfies all conditions in $Theorem$ 2.1 . For infinitely many $elliptic\ curve\ E$ have different $j - invariants$ by the form of the conductors of E. This completes the proof of $Theorem$ 1.

IV. CONCLUSION

In this work, we confirm Goldfeld's conjecture for infinitely many elliptic curves. We rely on the modular curve and the binary Goldbach problem. Of course, many questions related to the rank of the Goldfeld conjecture remain unresolved and will require further research and effort. In future studies, we may find answers to these questions.

DECLARATION STATEMENT

As the article's author, I must verify the accuracy of the following information after aggregating input from all authors.

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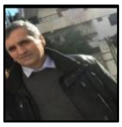

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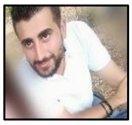

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