Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-6 Issue-1, April 2026

Every Sum of Two Positive Integers Has Either a Trivial or a Non-Trivial Common Factor

Joseph Wamukoya

***Abstract:** This paper investigates the arithmetic structure of exponential Diophantine equations of the form $A^x + B^y = k^n$, where $A, B, k, x, y, n \in Z^+$. Classical treatments such as the Beal Conjecture [1] and Fermat's Last Theorem (FLT) [2] restrict attention to exponents greater than two, leaving open the structural behavior of the equation for n = 1 and n = 2. This manuscript provides a unified framework addressing all positive integer exponents. A central theorem establishes that each term A^x, B^y, and k^n can be expressed as the sum of an arithmetic sequence whose number of terms and average term are positive integers, provided the equation has a trivial or non-trivial common factor. This elasticity property of k^n is derived through Gauss's method for summing arithmetic progressions. The case n = 2 recovers the classical identity for k^2 as the sum of the first k odd integers [3], revealing Pythagoras' theorem as a special instance of the general framework. For exponents exceeding two, if gcd(A, B,k) = 1, the arithmetic structure collapses, aligning with the Beal Conjecture as it is presented in the literature as a generalization of FLT [1]. The results demonstrate a consistent theory for all positive integer exponents and show that every sum of two positive integers has either a trivial or a non-trivial common factor.*

***Keywords:** Beal Conjecture; Arithmetic Sequences; Exponential Diophantine Equations; Fermat's Last Theorem; Pythagoras*

Nomenclature:
FLT: Fermat's Last Theorem

I. INTRODUCTION

The Beal Conjecture [1] and Fermat's Last Theorem (FLT) [2] both concern exponential Diophantine equations of the form

$$A^x + B^y = k^n \quad \dots \quad (1)$$

where A, B, k, x, y, n ∈ Z+ and the exponents satisfy x, y, n > 2. These classical formulations exclude the cases n = 1 and n = 2. This work examines all positive integer exponents and shows that a coherent arithmetic structure applies across all cases. The representation of k^2 as the sum of the first k odd integers [3] naturally connects with the general theorem presented here, revealing Pythagoras' identity as a special case within a broader arithmetic framework.

Manuscript received on 14 November 2025 | First Revised Manuscript received on 24 November 2025 | Second Revised Manuscript received on 16 March 2026 | Manuscript Accepted on 15 April 2026 | Manuscript published on 30 April 2026.

*Correspondence Author(s)

Joseph Kongani Wamukoya*, Department of Math/Physics, Nairobi, Westlands, Nairobi, Kenya. Email ID: wamukoyajoseph353@gmail.com, ORCID ID: 0009-0008-4325-1076

II. ELASTICITY THEOREM

Theorem 1 (Elasticity of a Number). For positive integers k, n, the value k^n can be expressed as the sum of an arithmetic sequence with mk terms, first term

$$\frac{k^{n-1}}{m} - mk + 1,$$

average term

$$\frac{k^{n-1}}{m},$$

and common difference 2, provided

$$m \in Q^+,\ mk \in Z^+, \qquad \frac{k^{n-1}}{m} \in \mathbb{Z}^+. \quad \dots \quad (2)$$

Proof.

$$k^n = \frac{mk}{2}\left[2\left(\frac{k^{n-1}}{m} - mk + 2\right) + (mk-1)\cdot 2\right] \quad \dots \quad (3)$$

$$k^n = mk\left(\frac{k^{n-1}}{m}\right),$$

$$k^n = k^n. \quad \dots \quad (4)$$

Consider the equation

$$A^x + B^y = k^n. \quad \dots \quad (5)$$

Multiplying the right-hand side of (3) by

$$\frac{mk}{mk} \quad \dots \quad (6)$$

converts k^n into an arithmetic sequence with mk terms and average $\frac{k^{n-1}}{m}$.

Set $n = 1$ in (5):

$$(A^x + B^y)1 = k(1). \quad \dots \quad (7)$$

Thus

$$A^x + B^y = k. \quad \dots \quad (8)$$

Set n = 2 in (5). A classical identity [3] states:

$$k^2 = 1 + 3 + 5 + \cdots + (2k - 1). \quad \dots \quad (9)$$

Hence

$$A^x + B^y = 1 + 3 + 5 + \cdots + (2k - 1). \quad \dots \quad (10)$$

This decomposes as

$$1+3+5+\cdots+(2t-1)+\frac{k-t}{2}[2(2k-1)+(k-t-1)(-2)] \quad \ldots \quad (11)$$

The first sequence sums to t^2, with average

$$\frac{t^2}{t}=t. \quad \ldots \quad (12)$$

The remaining $(k-t)$ terms sum to k^2-t^2, with average

$$\frac{k^2-t^2}{k-t}=k+t. \quad \ldots \quad (13)$$

Example. Consider the Pythagorean triple $8^2+6^2=10^2$. The value 8^2 has equivalent representations $64 = 4^3 = 2^6$. Each equivalent form corresponds to a different choice of m in the elasticity theorem, illustrating that the quantity can take multiple exponential forms depending on m. This demonstrates the representation's elastic nature.

Now let $n > 2$ in (5) with $\gcd(A, B, k) = 1$. Then:

$$A^x+B^y=mk\left(\frac{k^{n-1}}{m}\right). \quad \ldots \quad (14)$$

Convert both sides:

$$k\left(\frac{A^x}{k}\right)+k\left(\frac{B^y}{k}\right)=k(k^{n-1}) \quad . \ \ldots \quad (15)$$

The expressions inside the parentheses on the left are not integers when $\gcd(A, B, k) = 1$. Therefore, (15) cannot represent an arithmetic sequence with an integer average. This obstruction aligns with the Beal Conjecture, which asserts that if

$$A^x+B^y=C^z$$

With $x, y, z > 2$ have a solution in positive integers, then A, B, and C must share a common prime factor [1].

III. CONCLUSION

If the equation

$$A^x+B^y=k^n.$$

Holds in such a way that the triple (A, B, k) possesses either a trivial or a non-trivial common factor. Each of the quantities A^x, B^y, and k^n can be written as an arithmetic sequence whose number of terms and whose average term are positive integers. Consequently, whenever an equation of the form $A^x + B^y = k^n$ is satisfied under such conditions, the resulting sum necessarily reflects the presence of either a trivial or a non-trivial common factor among the bases involved.

DECLARATION STATEMENT

The references [1], [2], and [3] are more than ten years old. They are retained because they constitute foundational sources essential to the Beal Conjecture, Fermat's Last Theorem, and the classical identity for expressing k^2 as a sum of odd integers.

I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted with objectivity and without any external influence.
- **Ethical Approval and Consent to Participate:** The content of this article does not necessitate ethical approval or consent to participate with supporting documentation.
- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed solely.

REFERENCES

1. R. D. Mauldin, "A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem," Notices of the American Mathematical Society, vol. 44, no. 11, pp. 1436– 1439, 1997. Available at: https://www.ams.org/notices/199711/beal.pdf. The works remain significant, see the declaration
2. A. Wiles, "Modular elliptic curves and Fermat's last theorem," Annals of Mathematics, vol. 141, no. 3, pp. 443–551, 1995. DOI: https://doi.org/10.2307/2118559. Works remain significant, see the declaration
3. D. M. Burton, The History of Mathematics: An Introduction, 7th ed. McGraw–Hill, New York, 2010. ISBN 978-0-07-3383156. Works remain significant, see the declaration

AUTHOR'S PROFILE

Joseph Kongani Wamukoya is a physicist and mathematics researcher whose work focuses on number theory and the fundamental structure of physical laws. He is the author of the peer-reviewed article "On the Quantisation of Time, Space and Gravity" (Journal of Modern Physics, 2020, DOI: 10.4236/jmp. 2020.1111112), in which he develops a unified mathematical framework linking temporal, spatial, and gravitational behaviour. Wamukoya has taught physics and mathematics at the secondary and pre-university levels, including at the British School of Lomé in Togo, where he supported students in advanced scientific study. He continues to pursue research that strengthens the connection between mathematical reasoning and physical theory while contributing to accessible science education.