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Completed Functor $\widehat{S^{-1}}()$ of the Localization Functor $S^{-1}()$, Isomorphism and Adjunction

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***Abstract:** This article serves as a continuation of our previous work 1, which remains our primary reference for investigating specific homological properties with completion. Let the rings not be necessarily commutative and the modules be the unitary left (resp. right) modules. Let $(G,(G_n)_{n\in\mathbb{N}})$ be a filtered normal group equipped with the group topology associated with the filtration $(G_n)_{n\in\mathbb{N}}$ formed of normal subgroups and $\mathcal{C}(G)$ the set of Cauchy sequences with values in G. We define an equivalence relation $\mathcal{R}$ on $\mathcal{C}(G)$ by: $(x_n)\mathcal{R}(y_n) \Leftrightarrow (x_n)-(y_n)=(x_n-y_n)$ converges to 0, noted by $(x_n-y_n)\to 0$. The quotient set $\mathcal{C}(G)/\mathcal{R} := \{\widehat{(x_n)} \mid (x_n)\in\mathcal{C}(G)\}$ denoted $\widehat{G}$ is equipped with a group structure and is called the completed groupe of G. For any filtered ring (resp. left A-module) $(A,(I_n)_{n\in\mathbb{N}})$ (resp. $(M,(M_n)_{n\in\mathbb{N}})$), the completed group $\widehat{A}$ (resp. $\widehat{M}$) is equipped with a ring structure (resp. $\widehat{A}$-module) by $\widehat{(a_n)}\,\widehat{\times}\,\widehat{(b_n)} = \widehat{(a_nb_n)}\left(resp. \widehat{(a_n)}\cdot\widehat{(m_n)} = \widehat{(a_n\cdot m_n)}\right)$ where $\widehat{(a_n)},\widehat{(b_n)}\in\widehat{A}$ (resp. $\widehat{(m_n)}\in\widehat{M}$) called completed ring (resp. module) of A (resp. M). And for all saturated multiplicative subset S of A that satisfies the left Ore conditions, $\widehat{S}=\{\widehat{(x_n)}\in\widehat{A} \mid \widehat{(x_n)}\neq\widehat{0}$ and $\exists n_0\in\mathbb{N}, n\geq n_0, x_n\in S\}$ is a saturated multiplicative subset of $\widehat{A}$ that satisfies the left Ore conditions 1. Among the main results of this article, we have : - the functors $\widehat{S^{-1}}()$ is isomorphic to $\widehat{S}^{-1}(\widehat{A})\otimes_{\widehat{A}}-$. and $\widehat{S}^{-1}()$ is isomorphic to $\widehat{S^{-1}(A)}\otimes_{\widehat{A}}-$. - the functors $Hom_{\widehat{A}}(\widehat{S^{-1}A}\otimes_{\widehat{A}}\widehat{M},-)$ and $Hom_{\widehat{A}}(\widehat{S^{-1}A\otimes_A M},-)$ are isomorphic. - the functors $\widehat{S^{-1}A\otimes_A}$ - and $Hom_{\widehat{A}}(\widehat{S}^{-1}\widehat{A},-)$ are adjoints.*

This Study Allows How Establish a Relationship Between Completion [2] and Localization [4] Under the Assumptions of a Topological Structure.

***Keywords:** Ring, Modules, Filtration, Completion, Ore Condition, Localization, Isomorphisms, Categories, Functors, Completed Functor, Adjunction.*

I. INTRODUCTION

This article serves as a continuation of our previous work [1], which remains our primary reference for investigating specific homological properties with completion. In this paper, the rings are not necessarily commutative and the modules are the unitary left (resp. right) modules. Let $(G,(G_n)_{n\in\mathbb{N}})$ be a filtered normal group equipped with the group topology associated with the filtration $(G_n)_{n\in\mathbb{N}}$ formed of normal subgroups and $\mathcal{C}(G)$ the set of Cauchy sequences with values in G. We define an equivalence relation $\mathcal{R}$ on $\mathcal{C}(G)$ by: $(x_n)\mathcal{R}(y_n) \Leftrightarrow (x_n)-(y_n)=$ (x_n-y_n) converges to 0, noted by $(x_n-y_n)\to 0$. The quotient set $\mathcal{C}(G)/\mathcal{R} := \{\widehat{(x_n)} \mid (x_n)\in\mathcal{C}(G)\}$ denoted $\widehat{G}$ is equipped with a group structure and is called the completed groupe of G. For any filtered ring $(A,(I_n)_{n\in\mathbb{N}})$ (resp. left A-module $(M,(M_n)_{n\in\mathbb{N}})$), the completed group $\widehat{A}$ (resp. $\widehat{M}$) is equipped with a ring structure (resp. left $\widehat{A}$-module) by $\widehat{(a_n)}\,\widehat{\times}\,\widehat{(b_n)} = \widehat{(a_nb_n)}($ resp. $\widehat{(a_n)}\cdot\widehat{(m_n)} = \widehat{(a_n\cdot m_n)})$ where $\widehat{(a_n)},\widehat{(b_n)}\in\widehat{A}($ resp. $\widehat{(m_n)}\in\widehat{M})$ called completed ring (resp. module) of A(resp. M).

In the commutative case, the localization functor $S^{-1}()$ and the functor $S^{-1}A\otimes_A$ - have been studied by many authors [3]. However, in the non-commutative case, these functors have been addressed by few authors [4]. But, the completed functors $\widehat{S^{-1}}()$ and $\widehat{S^{-1}(A)}\otimes_{\widehat{A}}$ - have not been explicitly studied in either the commutative or non-commutative case to our knowledge, which constitutes the main objective of this work. In this article, we study the completed functor $\widehat{S^{-1}}()$ of the localization functor $S^{-1}()$ and the localization functor $\widehat{S}^{-1}()$, then their relationships with the tensor product functors $\widehat{S^{-1}(A)}\otimes_{\widehat{A}}-, \widehat{S^{-1}(A)\otimes_A}-, \widehat{S}^{-1}\widehat{A}\otimes_{\widehat{A}}-$ and $\mathrm{Hom}_{\widehat{A}}(\widehat{S^{-1}A}\otimes_{\widehat{A}}\widehat{M},-)$. We also study the adjunction between the functors and $\widehat{S^{-1}}()$, where S is a saturated multiplicative subset of A that satisfies the left Ore conditions and $\widehat{S}=\{\widehat{(x_n)}\in\widehat{A} \mid \widehat{(x_n)}\neq\widehat{0}$ and $\exists n_0\in\mathbb{N}, n\geq n_0, x_n\in S\}$ is the set of classes of Cauchy sequences in A with values in S that do not converge to 0 , which is a saturated multiplicative subset of $\widehat{A}$ that satisfies the left Ore conditions [1]. Thus, the main results in this article are: The section 1 consists of preliminary results. In section 2, we proove that:

- The completion functor $\widehat{S^{-1}}()$ of the functor $S^{-1}()$ is isomorphic to the localization functor $\widehat{S}^{-1}()$.
- The functors $\widehat{S}^{-1}()$ and $\widehat{S^{-1}A\otimes_A}-$ are isomorphic.
- The functors $\widehat{S}^{-1}()$ and $\widehat{S}^{-1}(\widehat{A})\otimes_{\widehat{A}}$ - are isomorphic.
- The functor $\widehat{S^{-1}}()$ is isomorphic to the functor $\widehat{S}^{-1}(\widehat{A})\otimes_{\widehat{A}}$-.
- The functor $\widehat{S}^{-1}(\widehat{A})\otimes_{\widehat{A}}$ - is isomorphic to $\widehat{S^{-1}A\otimes_A}-$.

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- The functor $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A} \otimes_{\hat{A}} \hat{M}, -)$ is isomorphic to the functor $\mathrm{Hom}_{\hat{A}}(S^{-1}\widehat{A \otimes_A} M, -)$.
- The functor $S^{-1}\widehat{A \otimes_A} -$ is adjoint to the functor $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, -)$.

II. DEFINITIONS AND PRELIMINARY RESULTS

Proposition 1. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring, $(M, (M_n)_{n\in\mathbb{N}})$ a filtered left A-module and $(N, (N_n)_{n\in\mathbb{N}})$ a filtered right A-module. Then, the sequence of subgroups with general term $(M \otimes_A N)_n = \sum_{i+j\le n} M_i \otimes_A N_j$ equips the tensor product group $M \otimes_A N$ with a filtered group structure.

Proof. $(M \otimes_A N)_{n\in\mathbb{N}}$ is a sequence of subgroups of $M \otimes_A N$ by definition.

Since $(M \otimes_A N)_{n+1} = \sum_{i+j\le n+1} M_i \otimes_A N_j = \sum_{i\le n+1} M_i \otimes_A N_0 + \sum_{i+j\le n+1} M_i \otimes_A N_{j+1}$ and $(M \otimes_A N)_n = \sum_{i+j\le n} M_i \otimes_R N_j = \sum_{i\le n} M_i \otimes_A N_0 + \sum_{i+j\le n} M_i \otimes_A N_{j-1}$,then $(M \otimes_A N)_n \subseteq (M \otimes_A N)_{n+1}$.

Theorem 1. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring, $(M, (M_n)_{n\in\mathbb{N}})$ a filtered left A module and $(N, (N_n)_{n\in\mathbb{N}})$ a filtered right A-module. Then, the tensor product group $M \otimes_A N$ is equipped with a topological group structure associated with the filtration $(M \otimes_A N)_{n\in\mathbb{N}}$.

Proof. [1]

Proposition 2. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered duo-ring and $(M, (M_n)_{n\in\mathbb{N}})$ a filtered left A-module. Then, the sequence of submodules with general term $(A \otimes_A M)_n = \sum_{i+j\le n} I_i \otimes M_j$ equips the left tensor product A-module $A \otimes_A M$ with a filtered module structure.

Proof. [2]

According to proposition 1, $(A \otimes_A M)_{n\in\mathbb{N}}$ is a filtration of $(A \otimes_A M)$ as a group when considering A as an A-module. Let $y \in I_n(A \otimes_A M)_m$, then there is $x_l \in I_n, u_{li} \in I_i$ and $v_{lj} \in M_j$, where $l \in J$ is a set of indices, such that

$$y = \sum_{l\in J} x_l \sum_{i+j\le m} u_{li} \otimes v_{lj} = \sum_{l\in J}\left(\sum_{i+j\le m} (x_l u_{li}) \otimes v_{lj}\right) = \sum_{i+j\le m}\left(\sum_{l\in J} (x_l u_{li}) \otimes v_{lj}\right)$$

$$= \sum_{i+j+n\le n+m}\left(\sum_{l\in J} (x_l u_{li}) \otimes v_{lj}\right)$$

$$= \sum_{i+k\le n+m}\left(\sum_{l\in J} (x_l u_{li}) \otimes v_{l(k-n)}\right), \text{ where } j = k-n \text{ and } k \ge n$$

$$= \sum_{i+k\le n+m}\left(\sum_{l\in J} (x_l u_{li}) \otimes v_{l(k-n)}\right), \text{ where } j = k-n \text{ and } k \ge n$$

$$= \sum_{i+k\le n+m}\left(\sum_{l\in J} x_l u_{li}\right) \otimes \left(\sum_{l\in J} v_{l(k-n)}\right)$$

$$= \sum_{i+k\le n+m} w_i \otimes z_k \text{ where } w_i = \sum_{l\in J} x_l u_{li} \in I_i \text{ and } z_k = \sum_{l\in J} v_{l(k-n)} \in M_k$$

$$= \sum_{i+k\le n+m} w_i \otimes z_k \in \sum_{i+k\le n+m} I_i \otimes M_k = (A \otimes A_A M)_{n+m}.$$

Then $I_n(A \otimes_A M)_m \subseteq (A \otimes_A M)_{n+m}, \forall n, m \in \mathbb{N}$.

Theorem 2. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the ring topology associated with $(I_n)_{n\in\mathbb{N}}$ and $(M, (I_n M)_{n\in\mathbb{N}})$ a filtered left A-module. Then, the tensor product A-module $A \otimes_A M$ is equipped with a topological module structure associated with the filtration $(A \otimes_A M)_{n\in\mathbb{N}}$.

Proof.

Let's show that the application $\cdot : A \times (A \otimes_A M) \to A \otimes_A M$ such that $(a, (b \otimes x)) \mapsto a(b \otimes x)$ is continuous. For all $0 \le i, j \le n \in \mathbb{N}$, we have:

$I_i \otimes I_j M \cong I_j M \Rightarrow \sum_{i+j\le n} I_i \otimes I_j M \cong \sum_{j\le n-i, i\le n} I_j M \Rightarrow (A \otimes_A M)_n \cong \sum_{j\le n-i, i\le n} I_j M$, then for all open set K of $A \otimes_A M$, there is an open set K' of M such that $\sum_{\substack{j=n-i \\ i\le n}} I_j M \subseteq K' \Leftrightarrow (A \otimes_A M)_n \subseteq K$.

Since $(a + I_n) \times (b \otimes x + (A \otimes_A M))$ is a neighborhood of all $(a, b \otimes x) \in \cdot^{-1}(K)$, then

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$$(a+I_n)\left(x+\sum_{\substack{j\leq n-i\\ i\leq n}} I_jM\right)=ax+I_nx+a\sum_{j\leq n-i,i\leq n} I_jM+I_n\sum_{j\leq n-i,i\leq n} I_jM$$

$$=ax+I_nx+\sum_{j\leq n-i,i\leq n} I_jM, \text{ since } I_j \text{ are ideals}$$

$$=ax+I_nx+I_nM+\sum_{j\leq n-i,1\leq i\leq n} I_jM=ax+\sum_{j\leq n-i,i\leq n} I_jM, \text{ since } I_nx\subset I_nM$$

$$=ax+\sum_{j\leq n-i,i\leq n} I_jM\subseteq K' \text{ since the laws of } M \text{ are continuous}$$

$$\Rightarrow (a+I_n)\left(x+\sum_{j\leq n-i,} I_jM\right)=ax+\sum_{j\leq n-i,i\leq n} I_jM\subseteq K'$$

$$\Rightarrow\cdot\left((a+I_n)\times(b\otimes x+(A\otimes_A M)_n)\right)=(a+I_n)(b\otimes x+(A\otimes_A M)_n)\subseteq K$$

$$\Rightarrow (a+I_n)\times(b\otimes x+(A\otimes_A M)_n)\ \subseteq .^{-1}(K).$$

Remark 1. We denote by $A\widehat{\otimes_A}M$ the completed of the left tensor product A module $A\otimes_A M$ equipped with the topology associated with the filtration $(A\otimes_A M)_{n\in\mathbb{N}}$ [1] .

Lemma 1. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed, S a saturated multiplicative subset of A that satisfies the left Ore conditions and $\hat{S}$ the set of classes of Cauchy sequences in A with values on S that do not converge to 0 . Then, $\hat{S}$ is a saturated multiplicative subset of $\hat{A}$ that satisfies the left Ore conditions.

Proof. [1]

Definition 1. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, S$ a saturated multiplicative subset of A that satisfies the left Ore conditions, $S^{-1}()$ the localization functor [4] and $\hat{F}()$ the completion functor. We call the completed functor of $S^{-1}()$ the functor $\widehat{S^{-1}}()=\hat{F}\circ S^{-1}()$ defined by $\widehat{S^{-1}}()=\hat{F}\circ S^{-1}(): AF-Mod\rightarrow \widehat{S^{-1}A}-$ Mod such that:

(1) For all $M\in Ob(AF-Mod)$, then

$$\widehat{S^{-1}(M)}=\hat{F}\circ S^{-1}(M)\in Ob\left(\widehat{S^{-1}A}F-Mod\right)$$

(2) For all $f\in \mathrm{Hom}_{AF}(M,M')$, then

$$\widehat{S^{-1}(f)}=\hat{F}\circ S^{-1}(f)\in \mathrm{Hom}_{\widehat{S^{-1}A}F}\left(\widehat{S^{-1}M},\widehat{S^{-1}M'}\right)$$

Proposition 3. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, S$ a saturated multiplicative subset of A that satisfies the left Ore conditions. Then, the functor $\widehat{S^{-1}}()$ is covariant, additive and left exact.

Proof. Use the functors completion $\hat{F}$ and $S^{-1}()$ [4].

III. FUNCTORIAL ISOMORPHISM AND ADJUNCTION

Theorem 3. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed. Then, the completed functor $\widehat{S^{-1}}()$ of the functor $S^{-1}()$ is isomorphic to the localization functor $\hat{S}^{-1}()$ (noted by $\hat{S}^{-1}()\cong \widehat{S^{-1}}()$).

Proof.

(1) $\vartheta_M:\widehat{S^{-1}M}\rightarrow \hat{S}^{-1}\hat{M}, \widehat{\left(\frac{m_n}{s_n}\right)}\mapsto\frac{\widehat{(m_n)}}{\widehat{(s_n)}}$ is isomorphic by the theorem 9.6 [1].

(2) Let $(M',(M'_n)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the group topology associated with $(M'_n)_{n\in\mathbb{N}}$ and $f:M\rightarrow M'$ a compatible modules morphism. Then, we have : $\hat{S}^{-1}(\hat{f}):\hat{S}^{-1}\hat{M}\rightarrow \hat{S}^{-1}\widehat{M'}, \widehat{\left(\frac{m_n}{s_n}\right)}\mapsto\frac{\widehat{(f(m_n))}}{\widehat{(s_n)}}$ and $\widehat{S^{-1}(f)}:\widehat{S^{-1}M}\rightarrow \widehat{S^{-1}M'}, \widehat{\left(\frac{m_n}{s_n}\right)}\mapsto\widehat{\left(\frac{f(m_n)}{s_n}\right)}$ are modules morphisms. We have the commutative diagramm :

$$\begin{array}{ccc}\widehat{S^{-1}M} & \xrightarrow{\vartheta_M} & \hat{S}^{-1}\widehat{M}\\ \widehat{S^{-1}(f)}\downarrow & & \downarrow \hat{S}^{-1}(\hat{f})\\ \widehat{S^{-1}M'} & \xrightarrow{\vartheta_{M'}} & \hat{S}^{-1}\widehat{M'}.\end{array}$$

Indeed :

$$\text{(a) } \hat{S}^{-1}(\hat{f})\circ\vartheta_M\left(\widehat{\left(\frac{m_n}{s_n}\right)}\right)=\hat{S}^{-1}(\hat{f})\left(\frac{\widehat{(m_n)}}{\widehat{(s_n)}}\right)=\frac{\widehat{(f(m_n))}}{\widehat{(s_n)}}$$

$$\text{(b) } \vartheta_{M'}\circ\widehat{S^{-1}(f)}\left(\widehat{\left(\frac{m_n}{s_n}\right)}\right)=\vartheta_{M'}\left(\widehat{\left(\frac{f(m_n)}{s_n}\right)}\right)=\frac{\widehat{(f(m_n))}}{\widehat{(s_n)}}$$

Proposition 4. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed, S a saturated multiplicative subset of A that satisfies the left Ore conditions, $(M,(M_n)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the group topology associated with $(M_n)_{n\in\mathbb{N}}$. Then, the correspondence defined by:

$$\begin{array}{llll}\phi: & \widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M} & \longrightarrow & \widehat{S^{-1}(A)\otimes_A M}\\ & \sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})} & \longmapsto & \phi\left(\sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)=\widehat{\left(\sum_{i\in I}\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\end{array}$$

is an isomorphism of left $\hat{A}$-modules.

Proof.

(1) Show that ϕ is a well-defined map.

$$\text{Let } \sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})},\ \sum_i\widehat{\left(\frac{b_{ni}}{t_{ni}}\right)}\otimes\widehat{(m'_{ni})}\in\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M} \text{ such that}$$

$$\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}=\sum_i\widehat{\left(\frac{b_{ni}}{t_{ni}}\right)}\otimes\widehat{(m'_{ni})}.$$

We have: $\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}=\sum_i\widehat{\left(\frac{b_{ni}}{t_{ni}}\right)}\otimes\widehat{(m'_{ni})}\Rightarrow\begin{cases}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}=\widehat{\left(\frac{b_{ni}}{t_{ni}}\right)}\\ \widehat{(m_{ni})}=\widehat{(m'_{ni})}\end{cases}$, $\forall i$ by definition of tensor product. Therefore, we have $\begin{cases}\left(\frac{a_{ni}}{s_{ni}}-\frac{b_{ni}}{t_{ni}}\right)\to 0\\ (m_{ni}-m'_{ni})\to 0\end{cases}$, $\forall i$. Moreover, $\left(\frac{a_{ni}}{s_{ni}}\otimes m_{n_i}-\frac{b_{ni}}{t_{ni}}\otimes m'_{n_i}\right)=\left(\frac{a_{ni}}{s_{ni}}-\frac{b_{ni}}{t_{ni}}\right)\otimes(m_{ni}-m'_{ni})-\left(\frac{a_{ni}}{s_{ni}}\right)\otimes(m'_{ni}-m_{ni})+\left(\frac{a_{ni}}{s_{ni}}-\frac{b_{ni}}{t_{ni}}\right)\otimes(m'_{ni})$ thus, $\left(\frac{a_{ni}}{s_{ni}}\otimes m_{n_i}-\frac{b_{ni}}{t_{ni}}\otimes m'_{n_i}\right)\to 0\forall i$,

$\Rightarrow\left(\sum_i\frac{a_{ni}}{s_{ni}}\otimes m_{n_i}-\sum_i\frac{b_{ni}}{t_{ni}}\otimes m'_{n_i}\right)\to 0.$

Therefore, $\widehat{\left(\sum_i\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}=\widehat{\left(\sum_i\frac{b_{ni}}{t_{ni}}\otimes m'_{ni}\right)}\Rightarrow\phi\left(\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)=\phi\left(\sum_i\widehat{\left(\frac{b_{ni}}{t_{ni}}\right)}\otimes\widehat{(m'_{ni})}\right)$.

(2) Show that ϕ is a morphism of $\hat{A}$-modules.

(a) It is clear that ϕ is a group morphism.

(b) Let $\widehat{(a_n)}\in\hat{A}$ and $\sum_i\widehat{\left(\frac{b_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\in\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}$, we have:

$$\begin{aligned}\phi\left(\widehat{(a_n)}\hat{\times}\sum_i\widehat{\left(\frac{b_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)&=\phi\left(\sum_i\widehat{(a_n)}\hat{\times}\widehat{\left(\frac{b_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)\\&=\phi\left(\sum_i\widehat{\left(a_n\times\frac{b_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)\\&=\widehat{\left(\sum_i a_n\times\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\\&=\widehat{(a_n)}\hat{\times}\widehat{\left(\sum_i\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\\&=\widehat{(a_n)}\hat{\times}\phi\left(\sum_{i\in I}\widehat{\left(\frac{b_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right).\end{aligned}$$

(3) Show that ϕ is bijective.

(a) The surjectivity is evident.

(b) Let $\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}$ and $\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m'_{ni})}\in\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}$ such that

$\phi\left(\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)=\phi\left(\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m'_{ni})}\right)$, let us show that

$\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}=\sum_i\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m'_{ni})}$.

We have:

$$\phi\left(\sum_i \widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m_{ni})}\right) = \phi\left(\sum_i \widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m'_{ni})}\right)$$

$$\Rightarrow \widehat{\left(\sum_i \frac{a_{ni}}{s_{ni}} \otimes m_{ni}\right)} = \widehat{\left(\sum_i \frac{a_{ni}}{s_{ni}} \otimes m'_{ni}\right)}$$

$\Rightarrow \left(\sum_i \frac{a_{ni}}{s_{ni}} \otimes m_{ni} - \sum_i \frac{a_{ni}}{s_{ni}} \otimes m'_{ni}\right) \to 0.$

Moreover, for all $i \in I, \sum_i \frac{a_{ni}}{s_{ni}} \otimes m_{ni} = \frac{1}{s_n} \otimes \sum_i x_{ni} a_{ni} m_{ni}$ and $\sum_i \frac{a_{ni}}{s_{ni}} \otimes m'_{ni} = \frac{1}{s_n} \otimes \sum_i x_{ni} a_{ni} m'_{ni}, \forall n \in \mathbb{N}$ where $s_n = \prod_i s_{ni}$ and $x_{ni} = s_n^{-1} s_{ni} \in S$ 。

Thus, $\left(\sum_i \frac{a_{ni}}{s_{ni}} \otimes m_{ni} - \sum_i \frac{a_{ni}}{s_{ni}} \otimes m'_{ni}\right) = \left(\frac{1}{s_n} \otimes \sum_i x_{ni} a_{ni} m_{ni} - \frac{1}{s_n} \otimes \sum_i x_{ni} a_{ni} m'_{ni}\right) = \left(\frac{1}{s_n} \otimes \sum_i x_{ni} a_{ni} [m_{ni} - m'_{ni}]\right) \to 0$

$\Rightarrow (\sum_i x_{ni} a_{ni} [m_{ni} - m'_{ni}]) \to 0$

$\Rightarrow (\sum_{i \in I} x_{ni} a_{ni} [m_{ni} - m'_{ni}]) = \hat{0}$

$\Rightarrow \sum_i \widehat{(x_{ni})} \hat{\times} \widehat{(a_{ni})} \hat{\times} \widehat{(m_{ni})} = \sum_i \widehat{(x_{ni})} \hat{\times} \widehat{(a_{ni})} \hat{\times} \widehat{(m'_{ni})}.$

Therefore, for all $i \in I, \widehat{(x_{ni})} \times \widehat{(a_{ni})} \hat{\times} \widehat{(m_{ni})} = \widehat{(x_{ni})} \hat{\times} \widehat{(a_{ni})} \hat{\times} \widehat{(m'_{ni})}, \forall n \in \mathbb{N} \Rightarrow \widehat{(a_{ni})} \times \widehat{(m_{ni})} = \widehat{(a_{ni})} \times \widehat{(m'_{ni})}, \forall i.$

Thus, $\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m_{ni})} = \widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m'_{ni})}, \forall i$, then $\sum_i \widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m_{ni})} = \sum_i \widehat{\left(\frac{a_{ni}}{s_{ni}}\right)} \otimes \widehat{(m'_{ni})}.$

Corollary 1. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed ring and $(M, (M_n)_{n\in\mathbb{N}})$ a filtered left A module equipped with the group topology associated with $(M_n)_{n\in\mathbb{N}}$. Then, $\widehat{S^{-1}A} \otimes_{\hat{A}} \hat{M}$ and $S^{-1}\widehat{(A)} \otimes_A M$ are isomorphic.

Corollary 2. Let $(A, (P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed ring, S a set of regular elements of $A \setminus P$ and $(M, (P^n M)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the P-adic topology. Then, $\widehat{A_P} \otimes_{\hat{A}} \widehat{M_P}$ and $A_P \widehat{\otimes_A} M_P$ are isomorphic.

Proposition 5. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed ring. Then, $\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}$ is isomorphic to $S^{-1}\widehat{(A) \otimes_A} M$.

Proof.

According to theorem 3, we have $\hat{S}^{-1}\hat{A} \cong \widehat{S^{-1}(A)}$, then $\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M} \cong \widehat{S^{-1}(A)} \otimes_{\hat{A}} \hat{M} \cong S^{-1}\widehat{(A) \otimes_A} M$.
Corollary 3. Let $(A, (P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed duo-ring, S a set of regular elements of $A \setminus P, (M, (P^n M)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the P-adic topology. Then, $\hat{A}_{\hat{P}} \otimes_{\hat{A}} \hat{M}_{\hat{P}}$ is isomorphic to $\widehat{A_P} \otimes_A \widehat{M_P}$.

Proposition 6. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed ring. Then, the correspondence:

$$\chi_{\widehat{M}} : \begin{array}{ccc} \hat{S}^{-1}\hat{A} \times \widehat{M} & \longrightarrow & \hat{S}^{-1}\widehat{M} \\ \left(\frac{\widehat{(a_n)}}{\widehat{(s_n)}}, \widehat{(m_n)}\right) & \longmapsto & \frac{\widehat{(a_n)}\hat{\cdot}\widehat{(m_n)}}{\widehat{(s_n)}} \end{array} \text{ is } \hat{A}\text{-bilinear.}$$

Theorem 4. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then, the functors $\hat{S}^{-1}()$ and $\hat{S}^{-1}(\hat{A}) \otimes_{\hat{A}} -$ are isomorphic.

Proof. There is a left $\hat{A}$-module, according to proposition 6, defined by:

$$\overline{\chi}_{\widehat{M}} : \begin{array}{ccc} \hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \widehat{M} & \longrightarrow & \hat{S}^{-1}\widehat{M} \\ \sum_i \frac{\widehat{(a_{ni})}}{\widehat{(s_{ni})}} \otimes \widehat{(m_{ni})} & \longmapsto & \sum_i \frac{\widehat{(a_{ni})}\hat{\cdot}\widehat{(m_{ni})}}{\widehat{(s_{ni})}} \end{array}$$

according to the universal property of the tensor product. $\overline{\chi}_{\hat{M}}$ is indeed bijective, as follows:
(1) According to 4, $S^{-1}M$ and $S^{-1}A \otimes_A M$ are isomorphic. Consequently, $\widehat{S^{-1}M}$ and $S^{-1}\widehat{A \otimes_A} M$ are also isomorphic.

By the theorem 3, $\hat{S}^{-1}\hat{A} \otimes \hat{M}$ and $\widehat{S^{-1}A} \otimes_{\hat{A}} \hat{M}$ are isomorphic. Since $\widehat{S^{-1}A} \otimes_{\hat{A}} \hat{M}$ and $S^{-1}\widehat{(A) \otimes_A} M$ are isomorphic, by Proposition 4, then $\hat{S}^{-1}\hat{A} \otimes \hat{M}$ and $\hat{S}^{-1}\hat{M}$ are isomorphic.
Therefore, there exists $\omega_{\hat{M}} : \hat{S}^{-1}\hat{M} \longrightarrow \hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}$

(2) Consider $(M,(M_n)_{n\in\mathbb{N}})$ and $(M',(M'_n)_{n\in\mathbb{N}})$ two left A-modules, and $f: M \to M'$ a compatible module morphism. Then, according to point 1 , the following diagram commutes:

$$\begin{array}{ccc} \widehat{S}^{-1}\widehat{M} & \xrightarrow{\omega_{\widehat{M}}} & \widehat{S}^{-1}\widehat{A}\otimes_{\widehat{A}}\widehat{M} \\ \hat{f}\downarrow & & \downarrow 1_{\widehat{S}^{-1}\widehat{A}}\otimes\hat{f} \\ \widehat{S}^{-1}\widehat{M'} & \xrightarrow{\omega_{\widehat{M'}}} & \widehat{S}^{-1}\widehat{A}\otimes_{\widehat{A}}\widehat{M'}. \end{array}$$

Indeed:

Let $\frac{\widehat{(m_n)}}{\widehat{(s_n)}} \in \hat{S}^{-1}\widehat{M}$, then:

$$\cdot\, \omega_{\widehat{M}'} \circ \hat{S}^{-1}(\hat{f})\left(\frac{\widehat{(m_n)}}{\widehat{(s_n)}}\right) = \omega_{\widehat{M'}}\left(\frac{f\left(\widehat{(m_n)}\right)}{\widehat{(s_n)}}\right) = \omega_{\widehat{M'}}\left(\frac{\widehat{(f(m_n))}}{\widehat{(s_n)}}\right)$$
$$= \frac{\hat{1}}{\widehat{(s_n)}} \otimes \left(\widehat{f(m_n)}\right)$$
$$\cdot\left(1_{\hat{S}^{-1}\hat{A}} \otimes \hat{f}\right)\circ \omega_{\widehat{M'}}\left(\frac{\widehat{(m_n)}}{\widehat{(p_n)}}\right) = \left(1_{\hat{S}-1\hat{A}} \otimes \hat{f}\right)\left(\frac{\hat{1}}{\widehat{(s_n)}} \otimes \widehat{(m_n)}\right)$$
$$= \frac{\hat{1}}{\widehat{(s_n)}} \otimes \left(\widehat{f(m_n)}\right)$$

Thus, the functors $\hat{S}^{-1}()$ and $\widehat{S}^{-1}(\widehat{A}) \otimes_{\widehat{A}} -$ are isomorphic.

Corollary 4. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then, the functors $\hat{S}^{-1}()$ and $S^{-1}\widehat{A \otimes_A} -$ are isomorphic.

Proof.

By the theorem 3, we have $\hat{S}^{-1}() \cong \widehat{S^{-1}}() \cong S^{-1}\widehat{(A)\otimes_A}$.

Corollary 5. Let $(A,(P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed duo-ring, S a subset of regular elements of $A \setminus P$, $(M,(P^nM)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the P-adic topology. Then, the functors $\hat{S}^{-1}()$ and $A_P\widehat{\otimes_A} -$ are isomorphic.

Theorem 5. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then, we have:

(1) $\hat{S}^{-1}() \cong \widehat{S^{-1}(A)} \otimes_{\hat{A}}$.
(2) $\hat{S}^{-1}(\hat{A}) \otimes_{\hat{A}} - \cong S^{-1}\widehat{A\otimes_A} -$.
(3) $\widehat{S^{-1}}() \cong \hat{S}^{-1}(\hat{A}) \otimes_{\hat{A}} -$.

Proof.

(1) We have $\hat{S}^{-1}\hat{A}$ and $\widehat{S^{-1}A}$ are isomorphic (theorem 3). Thus, $\hat{S}^{-1}() \cong \hat{S}^{-1}\hat{A} \otimes_{\hat{A}} - \cong \widehat{S^{-1}(A)} \otimes_{\hat{A}} -$ by theorem 4 .
(2) It suffices to observe that $\widehat{S^{-1}}() \cong S^{-1}\widehat{A\otimes_A} - \cong \hat{S}^{-1}\hat{A} \otimes_{\hat{A}} - \cong \hat{S}^{-1}()$, by theorem 4.
(3) Follows from 1. and 2.

Corollary 6. Let $(A,(P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed ring, S a subset consisting of regular elements of $A \setminus P$, $(M,(P^nM)_{n\in\mathbb{N}})$ a filtered left A-module equipped with the P-adic topology, and $\hat{S} = \{\widehat{(x_n)} \in \hat{A} \mid \widehat{(x_n)} \neq \hat{0}$ and $\exists n_0 \in \mathbb{N}, n \geq n_0, x_n \in S\}$ the set of equivalence classes of Cauchy sequences in A with values in S that do not converge to 0 . Then, we have:

(1) $\hat{S}^{-1}() \cong \widehat{A_P} \otimes_{\hat{A}}$.
(2) $\hat{A}_{\hat{P}} \otimes_{\hat{A}} - \cong A_P\widehat{\otimes_A} -$
(3) $\widehat{S^{-1}}() \cong \hat{A}_{\hat{P}} \otimes_{\hat{A}} -$.

Proposition 7. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, and $(M,(M_n)_{n\in\mathbb{N}}),(M',(M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then, there is an isomorphism

$$\begin{array}{rcl} \psi_{M'}: \mathrm{Hom}_{\hat{A}}\left(\widehat{S^{-1}A} \otimes_{\hat{A}} \widehat{M}, \widehat{M'}\right) & \to & \mathrm{Hom}_{\hat{A}}\left(S^{-1}\widehat{A\otimes_A} M, \widehat{M'}\right) \\ f & \mapsto & \psi_{M'}(f): S^{-1}\widehat{A\otimes_A} M \longrightarrow \widehat{M'} \\ & & \left(\sum_{\iota\in I} \frac{\widehat{a_{n\iota}}}{s_{n\iota}} \otimes m_{n\iota}\right) \mapsto f\left(\sum_{\iota\in I}\left(\frac{\widehat{a_{n\iota}}}{s_{n\iota}}\right) \otimes \widehat{(m_{n\iota})}\right) \end{array}$$

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Proof.

It is evident that ψ is well-defined (existence) and is a morphism of $\hat{A}$-modules. Let us show that ψ is bijective. Consider the application

$$\begin{array}{llll} \psi'_{M'}: & Hom_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, \widehat{M'}) & \to & Hom_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'}) \\ & f & \mapsto & \psi'_{M'}(f): \widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M} \longrightarrow \widehat{M'} \\ & & & \sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})} \longmapsto f\left(\widehat{\left(\sum_{i\in I}\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\right) \end{array}$$

$\psi'_{M'}$ is a morphism of $\hat{A}$-modules.

For every $g \in \mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$, we have:

$$\psi'_{M'}\circ\psi_{M'}(g)\left(\widehat{\left(\sum_{i\in I}\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\right) = g\left(\widehat{\left(\sum_{i\in I}\frac{a_{ni}}{s_{ni}}\otimes m_{ni}\right)}\right)$$

$\Rightarrow \psi'_{M'}\circ\psi_{M'}(g) = g \Rightarrow \psi'_{M'}\circ\psi_{M'} = \mathrm{Id}_{\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M},\widehat{M'})}$.

Proposition 8. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, and $(M,(M_n)_{n\in\mathbb{N}}), (M',(M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then, $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$ are isomorphic.

Proof.

By the theorem 3, we have $\hat{S}^{-1}\hat{A} \cong \widehat{S^{-1}(A)}$, hence $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'}) \cong \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$ by the proposition 7.

Corollary 7. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, and $(M,(M_n)_{n\in\mathbb{N}}), (M',(M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then, $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, \widehat{M'})$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$ are isomorphic.

Corollary 8. Let $(A,(P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed duo-ring, S a subset of regular elements of $A\setminus P$, $(M,(P^nM)_{n\in\mathbb{N}})$ and $(M',(P^nM')_{n\in\mathbb{N}})$ two filtered left A-modules. Then, we have:

(1) $\mathrm{Hom}_{\hat{A}}(\widehat{A_P}\otimes_{\hat{A}}\widehat{M}, \widehat{M'}) \cong \mathrm{Hom}_{\hat{A}}(\widehat{A_P\otimes_A M}, \widehat{M'})$.

(2)$\mathrm{Hom}_{\hat{A}}(\widehat{A_P}\otimes_{\hat{A}}\widehat{M}, \widehat{M'}) \cong \mathrm{Hom}_{\hat{A}}(\hat{A}_{\hat{P}}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$.

(3) $\mathrm{Hom}_{\hat{A}}(\widehat{A_P\otimes_A M}, \widehat{M'}) \cong \mathrm{Hom}_{\hat{A}}(\hat{A}_{\hat{P}}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$.

Theorem 6. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, $(M,(M_n)_{n\in\mathbb{N}}), (M',(M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then, the functors $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, -)$ and $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, -)$ are isomorphic.

Proof.

Let $\widehat{M'}$ be the completion of a filtered left A-module $(M',(M'_n)_{n\in\mathbb{N}})$. According to proposition 7) $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$ and $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, \widehat{M'})$ are isomorphic.

Let $(M',(M'_n)_{n\in\mathbb{N}}), (M'',(M''_n)_{n\in\mathbb{N}})$ be two filtered left A-modules and $f: M' \to M''$ a compatible morphism of A-modules. Then, we have the following commutative diagram:

$$\begin{array}{ccc} Hom_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'}) & \xrightarrow{\hat{f}_*} & Hom_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M''}) \\ {\scriptstyle\psi_{M'}}\downarrow & & \downarrow{\scriptstyle\psi_{M''}} \\ Hom_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, \widehat{M'}) & \xrightarrow{\hat{f}^*} & Hom_{\hat{A}}(\widehat{S^{-1}A\otimes_A M}, \widehat{M''}). \end{array}$$

Indeed, for all $g \in \mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, \widehat{M'})$, we have:

- $\widehat{f^*}\circ\psi_{M'}(g) = \hat{f}\circ\psi_{M'}(g) = \hat{f}\circ g\left(\sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)$.
- $\psi_{M''}\circ\hat{f}_* = \psi_{M''}\circ(\hat{f}(g)) = (\hat{f}\circ g)\left(\sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right) = \hat{f}\circ g\left(\sum_{i\in I}\widehat{\left(\frac{a_{ni}}{s_{ni}}\right)}\otimes\widehat{(m_{ni})}\right)$.

Thus, $\hat{f}^*\circ\psi_{M'} = \psi_{M''}\circ\hat{f}_*$.

Corollary 9. Let $(A,(I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, $(M,(M_n)_{n\in\mathbb{N}}), (M',(M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then, the functors:

(1) $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}\otimes_{\hat{A}}\widehat{M}, -)$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}\otimes_{\hat{A}}\widehat{M}, -)$ are isomorphic.

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(2) $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}, -)$ and $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A \otimes_A M}, -)$ are isomorphic.

Proposition 9. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}, \hat{A}$ its completed ring, $(M, (M_n)_{n\in\mathbb{N}}), (M', (M'_n)_{n\in\mathbb{N}})$ two filtered left

A-modules. Then, $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}, \hat{M}')$ and $\mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, \hat{M}')\right)$ are isomorphic.

Proof. It suffices to show that the map defined by:

$$\phi_{MM'}: \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}, \hat{M}') \to \mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}}\hat{A}, \hat{M}')\right)$$
$$f \mapsto \phi_{MM'}(f): \hat{M} \to \mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}}\hat{A}, \hat{M}')$$
$$\widehat{(m_n)} \longmapsto \phi_{MM'}(f)(\widehat{(m_n)})$$

is an isomorphism of $\hat{A}$-modules, reasoning similarly as in Proposition 7

Corollary 10. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, $\hat{A}$ its completed ring, $(M, (M_n)_{n\in\mathbb{N}})$ and $(M', (M'_n)_{n\in\mathbb{N}})$ two filtered left A-modules. Then,

$\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{M}, \hat{M}')$ and $\mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, \hat{M}')\right)$ are isomorphic.

Proof.

Since $\hat{S}^{-1}\hat{M}$ and $\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}$ are isomorphic, $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{M}, \hat{M}')$ is isomorphic to $\mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, \hat{M}')\right)$ by the proposition 9

Corollary 11. Let $(A, (P_n)_{n\in\mathbb{N}})$ be a filtered duo-ring equipped with the P-adic topology, $\hat{A}$ its completed ring, S a subset of regular elements of $A \setminus P, (M, (P^n M)_{n\in\mathbb{N}})$

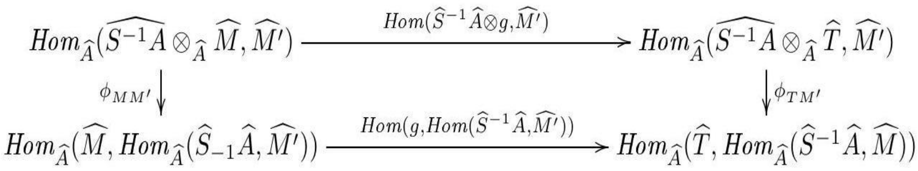
and $(M', (P^n M')_{n\in\mathbb{N}})$ two filtered left A-modules. Then, $\mathrm{Hom}_{\hat{A}}(\hat{A}_{\hat{P}} \otimes_{\hat{A}} \hat{M}, \hat{M}')$ and $\mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\hat{A}_{\hat{P}}, \hat{M}')\right)$ are isomorphic.

Theorem 7. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology defined by $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then, the functors $\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} -$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, -)$ are adjoints.

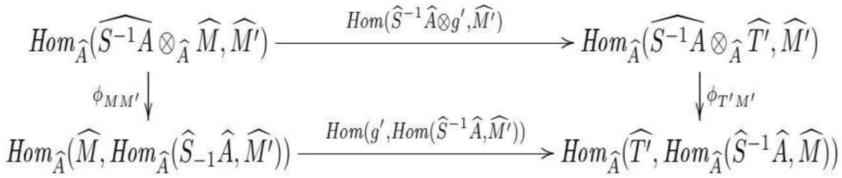
Proof.

Since $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A} \otimes_{\hat{A}} \hat{M}, \hat{M}') \cong \mathrm{Hom}_{\hat{A}}\left(\hat{M}, \mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, \hat{M}')\right)$ by the Proposition 9, it suffices to verify that:

(1) For every left $\hat{A}$-module $\hat{T}$ that is the completion of the filtered left A-module $(T, (T_n)_{n\in\mathbb{N}})$ and $g: \hat{T} \to \hat{M}$, the following diagram:



is commutative. (2) For every left $\hat{A}$-module $\widehat{T'}$ that is the completion of the filtered left A module $(T', (T'_n)_{n\in\mathbb{N}})$ and $g': \widehat{M'} \to \hat{T}$, the following diagram: is commutative [5].



Proposition 10. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology defined by $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then, the functors $\widehat{S^{-1}A \otimes_A} -$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, -)$ are adjoints.

Proof.

Since $\hat{S}^{-1}(\hat{A}) \otimes_{\hat{A}} - \widehat{S^{-1}A \otimes_A} -$, then, by the theorem γ, $\widehat{S^{-1}A \otimes_A} -$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, -)$ are adjoints.

Theorem 8. Let $(A, (I_n)_{n\in\mathbb{N}})$ be a filtered ring equipped with the group topology associated with $(I_n)_{n\in\mathbb{N}}$, and $\hat{A}$ its completed ring. Then:

(1) The functors $\widehat{S^{-1}}()$ and $\mathrm{Hom}_{\hat{A}}(\hat{S}^{-1}\hat{A}, -)$ are adjoints.

(2) The functors $\hat{S}^{-1}()$ and $\mathrm{Hom}_{\hat{A}}(\widehat{S^{-1}A}, -)$ are adjoints.

Proof.

(1) By the the theorem 5 and the proposition 10, we have the result.

(3) It is similar.

IV. CONCLUSION

This work is a continuation of our previous research on the link between the properties of topological completion and those of localization 1. We introduced the completion functor $\widehat{S^{-1}}()$ of the localization functor $S^{-1}()$1 and studied its relationships with the Hom functor, the tensor product functor $\widehat{S^{-1}A} \otimes_{\hat{A}}$-, as well as the completed functor $S^{-1}\widehat{A \otimes_A} -$ of the tensor product functor $S^{-1}A \otimes_A -$, among others.

The results obtained allowed us to highlight several functorial isomorphisms 5 and functorial adjunctions 7, 8. In particular, we have shown that the completion of a localization functor is naturally isomorphic to the localization of its topological group completion 3.

These results contribute to clarifing the structural relationships between completion and localization in a topological and in the non necessarily commutative case, and thus open new perspectives for the study of homological functor H_n properties, notably the derived functors Ext, Tor with the properties of completion in the non necessarily commutative case.

DECLARATION STATEMENT

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