

# Study of Some Degree Five Identities of Type (4,1)



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**Abstract:** In non-associative algebra, irreducible identities of degree five are the least studied. Following Osborn's studies, the only identity of type (5) has generated very little literature, as seen in ([1]) and ([2]). Hence, our interest in identities of the following type. The purpose of this study is to enable us to consider a baric case study at a later stage, such as in ([3]) and ([4]). This paper is devoted to the study of three of type (4,1), taken from the families of irreducible degree five identities of Osborn. We conduct this study in the presence of an idempotent, through a Peirce decomposition, depending on whether the Peirce polynomial is reducible or not over the base field of the algebra. In each studied case, we find two orthogonal subalgebras. Therefore, in the first two studied identities, we manage to show that there is a homomorphism over one of the subspaces of A, whose kernel is an ideal.

**Keywords:** Idempotent, Linearization, Peirce Decomposition.

## I. INTRODUCTION

J. M. Osborn has determined the whole family of irreducible degree five identities, not implied by commutativity: an identity is said to be irreducible if, despite the presence of a unity element, it is not a consequence of an identity of a lower degree. He finds, over a field F of characteristic not 2, 3 or 5, five parameterized identities, of which:

$$\beta_1 [yx^4 - 4(yx^3)x + 6((yx^2)x)x - 3(((yx)x)x)x] + \beta_2 [-y(x^2 \cdot x^2) + 5(yx^3)x - 9((yx^2)x)x + 4(((yx)x)x)x + ((yx)x^2)x + (yx^2)x^2 - (yx)x^3] + \beta_3 [-y(x^2 \cdot x^2) + 5(yx^3)x - 8((yx^2)x)x + 3((yx \cdot x)x)x + ((yx)x^2)x + (yx \cdot x)x^2 - (yx)x^3] = 0 \quad (1)$$

with  $\beta_1, \beta_2$  and  $\beta_3$  in F.

In fact, identity (1) is different from the one studied by J. M.

Osborn because of the factor of  $\beta_3$ , but it is the same, concerning the values of the parameters he has found. In our study, we are interested in the following identities.

$$yx^4 - 4(yx^3)x + 6((yx^2)x)x - 3(((yx)x)x)x = 0, \quad (2)$$

$$-y(x^2 \cdot x^2) + 5(yx^3)x - 9((yx^2)x)x + 4(((yx)x)x)x + ((yx)x^2)x + (yx^2)x^2 - (yx)x^3 = 0, \quad (3)$$

and

$$-y(x^2 \cdot x^2) + 5(yx^3)x - 8((yx^2)x)x + 3((yx \cdot x)x)x + ((yx)x^2)x + (yx \cdot x)x^2 - (yx)x^3 = 0, \quad (4)$$

which corresponds respectively to  $\beta_1 = 1, \beta_2 = \beta_3 = 0$ ;  $\beta_2 = 1, \beta_1 = \beta_3 = 0$ ;  $\beta_3 = 1, \beta_1 = \beta_2 = 0$ .

We assume that the studied algebras contain a nonzero idempotent e, and F is a commutative field of characteristic not 2, 3, 5 or 7. This will enable us to conduct our study as in ([2]).

## II. IDENTITY (2)

In this section, A is an algebra defined by identity (2). Setting  $x=e$  in (2), we have

$$[L_e(L_e - I)(3L_e^2 - 3L_e + I)](y) = 0, \quad (1.1)$$

with  $y \in A$  and  $L_e$  being the multiplication by e. Let's consider  $p(t) = -3t^4 + 6t^3 - 4t^2 + t$  which factorization is  $p(t) = -t(t-1)(3t^2 - 3t + 1)$ . We examine two cases:

### A. Polynomial $3t^2 - 3t + 1$ is Irreducible over F

In this case, the Peirce decomposition of A is  $A = A_0 \oplus A_1 \oplus B$ , with  $A_\mu = \{x \in A | ex = \mu x\}$  ( $\mu = 0, 1$ ) and  $B = Ker(3L_e^2 - 3L_e + I)$ .

The relations between the Peirce components are stated in the following theorem.

#### Theorem

Let A be an algebra defined by (2). Assume that  $3t^2 - 3t + 1$  is irreducible over  $F[t]$ . Then the Peirce decomposition of A is  $A = A_0 \oplus A_1 \oplus B$ , with  $A_0$  and  $A_1$  being orthogonal subalgebras,  $A_\lambda \subseteq B$  ( $\lambda = 0, 1$ ) and  $B^2 = 0$ . Moreover, we have:

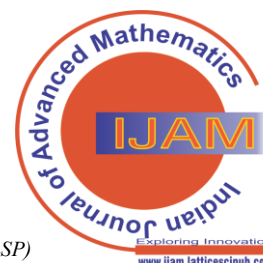
$$i. e(wx_\lambda) = (ew)x_\lambda$$

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ii.  $y_\lambda(x_\lambda w) = x_\lambda(y_\lambda w)$

iii.  $y_{1-\lambda}(x_\lambda w) = x_\lambda(y_{1-\lambda} w)$

iv.  $(L_e - I)(w(x_0 y_0)) = -y_0(x_0 w) = -x_0(y_0 w)$

v.  $e(w(x_1 y_1)) = y_1(x_1 w) = x_1(y_1 w)$

with  $w \in B$  and  $\lambda = 0, 1$ .

Proof

Linearizing identity (2), we have

$$y(zx^3) + y(x(zx^2)) + 2y(x(xz)) - 4z(yx^3) - 4x(y(zx^2)) - 8x(y(xz)) + 6z(x(yx^2)) + 6x(z(yx^2)) + 12x(x(yz)) - 3z(x(xz)) - 3x(z(xz)) - 3x(x(zz)) - 3x(x(zz)) = 0. \quad (1.2)$$

Let's consider  $x=e$  and simplify, we have

$$y(2L_e^3 + L_e^2 + L_e)(z) + z(-3L_e^3 + 6L_e^2 - 4L_e)(y) + (12L_e^2 - 4L_e)(y(ez)) + (-3L_e^2 + 6L_e)(z(ey)) - 8e(y(ez)) - 3e(z(ey)) - 3L_e^3(yz) = 0. \quad (1.3)$$

Let's take  $z \in A_\lambda$  and  $y \in B$  in equation (1.3). Since  $y \in B$ , we have  $3e(ey) = 3ey - y$  and  $3e(e(y)) = 2ey - y$ . Then

(1.3) becomes

$$[-3L_e^3 + 12\lambda L_e^2 + (-8\lambda^2 - 4\lambda + 1)L_e + (2\lambda^3 + \lambda^2 + \lambda - 1)I](yz) + (-3L_e^2 + 3L_e)(z(ey)) = 0. \quad (1.4)$$

In the following, for every  $t \in A$ , we shall note  $t = t_0 + t_1 + t_2$ , with  $t_i \in A_i$  ( $i=0,1$ ) and  $t_2 \in B$ , then (1.4) becomes

$$(2\lambda^3 + \lambda^2 + \lambda - 1)(yz)_0 + (2\lambda^3 - 7\lambda^2 + 9\lambda - 3)(yz)_1 + (-8\lambda^2 + \lambda - 1)e(yz)_2 + (2\lambda^3 + \lambda^2 - 3\lambda)(yz)_2 + [z(ey)]_2 = 0. \quad (1.5)$$

If  $\lambda=0$  in (1.5), then we have  $-(yz)_0 - 3(yz)_1 - e(yz)_2 + [z(ey)]_2 = 0$ , that is  $(yz)_0 = (yz)_1 = 0$  and  $e(yz)_2 = [z(ey)]_2$ . So we have  $yz \in B$  and  $e(yz)_2 = [z(ey)]_2$  which means  $e(yz) = (ey)z$ . And we finally have  $A_0 B \subseteq B$ .

If  $\lambda=1$  in (1.5), then we have  $3(yz)_0 + (yz)_1 - e(yz)_2 + [z(ey)]_2 = 0$ , which means  $(yz)_0 = (yz)_1 = 0$  and  $e(yz)_2 = [z(ey)]_2$ . This leads to  $yz \in B$  and  $e(yz)_2 = [z(ey)]_2$ , which means  $e(yz) = (ey)z$ . And we have  $A_1 B \subseteq B$ .

Now let's consider  $y, z \in B$  in (1.3). It gives, by simplifications,

$$(-3L_e^3 + \frac{11}{3}L_e - 2I)(yz) + (12L_e^2 - 12L_e + \frac{10}{3}I)(y(ez)) + (-3L_e^2 + 3L_e)(z(ey)) = 0, \quad (1.6)$$

switching  $y$  and  $z$  in (1.6) we have

$$(-3L_e^3 + \frac{11}{3}L_e - 2I)(zy) + (12L_e^2 - 12L_e + \frac{10}{3}I)(z(ey)) + (-3L_e^2 + 3L_e)(y(ez)) = 0. \quad (1.7)$$

Difference between (1.6) and (1.7) gives

$$(15L_e^2 - 15L_e + \frac{10}{3}I)(z(ey)) - (15L_e^2 - 15L_e + \frac{10}{3}I)(y(ez)) = 0,$$

which means  $(15L_e^2 - 15L_e + \frac{10}{3}I)(z(ey) - y(ez)) = 0$ .

Let's consider  $q(t) = 15t^2 - 15t + \frac{10}{3}$ . Since

$\gcd(p(t), q(t))=1$  then  $z(ey) - y(ez) = 0$ , that means  $z(ey) = y(ez)$ .

Using this last equation in (1.6), it gives

$$(-3L_e^3 + \frac{11}{3}L_e - 2I)(yz) + (9L_e^2 - 9L_e + \frac{10}{3}I)(z(ey)) = 0,$$

which means

$$-2(yz)_0 + \frac{10}{3}(z(ey))_0 - \frac{4}{3}(yz)_1 + \frac{10}{3}(z(ey))_1 + \frac{5}{3}e(yz)_2 - (yz)_2 + \frac{1}{3}(z(ey))_2 = 0.$$

That leads to

$$-2(yz)_0 + \frac{10}{3}(z(ey))_0 = 0, \quad -\frac{4}{3}(yz)_1 + \frac{10}{3}(z(ey))_1 = 0$$

and

$$\frac{5}{3}e(yz)_2 - (yz)_2 + \frac{1}{3}(z(ey))_2 = 0,$$

so, it finally gives

$$(yz)_0 = \frac{5}{3}(z(ey))_0, \quad (1.8)$$

$$(yz)_1 = \frac{5}{2}(z(ey))_1, \quad (1.9)$$

$$(z(ey))_2 = -5e(yz)_2 + 3(yz)_2. \quad (1.10)$$

Replacing  $y$  by  $ey$  in (1.8), we have  $(z(ey))_0 = \frac{5}{3}z(e(ey))_0$

$$= \frac{5}{3}(z(ey))_0 - \frac{5}{9}(yz)_0 = \frac{4}{9}(yz)_0$$

that means

$$(yz)_0 = \frac{9}{4}(z(ey))_0. \quad (1.11)$$

Difference between (1.8) and (1.11) gives  $\frac{7}{12}(z(ey))_0 = 0$ .

Since the characteristic of  $F$  is not

2, 3 or 7, we have  $(z(ey))_0 = 0$  so,  $(yz)_0 = 0$ .

Replacing  $y$  by  $ey$  in (1.9), we have

$$(z(ey))_1 = \frac{5}{2}z(e(ey))_1 = \frac{5}{2}(z(ey))_1 - \frac{5}{6}(yz)_1 = \frac{1}{6}(yz)_1$$

which means

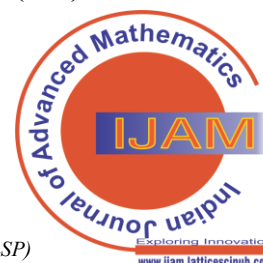
$$(yz)_1 = 6(z(ey))_1. \quad (1.12)$$

Subtracting (1.9) and (1.12), we have  $\frac{7}{12}(z(ey))_1 = 0$ , that means  $(z(ey))_1 = 0$  and  $(yz)_1 = 0$ .

Since  $(yz)_0 = (yz)_1 = 0$ , then we have  $yz \in B$ , that means  $B^2 \subseteq B$ . And (1.10) can be simply written

$$(z(ey)) = -5e(yz) + 3(yz). \quad (1.13)$$

Replacing  $y$  by  $ey$  in (1.13) and simplifying, we have



$(z(e_y)) = -5e(yz) + 4(yz)$ . (1.14)  
Subtracting (1.13) and (1.14) gives  $yz = 0$ . Finally, we have  $B^2 = 0$ .

Now, we study the products of three factors. Partially linearizing (1.2) gives

$$y(z(wx^2)) + 2y(z(x(wx))) + y(w(zx^2)) + 2y(x(z(xw))) + 2y(w(x(xz))) + 2y(x(w(xz))) + 2y(x(x(wz))) - 4z(y(wx^2)) - 8z(y(x(xw))) - 4w(y(zx^2) - 8x(y(z(xw))) - 8w(y(x(xz))) - 8x(y(w(xz))) - 8x(y(x(wz))) + 6z(w(yx^2)) + 12z(x(y(wx))) + 6w(z(yx^2)) + 12x(z(y(wx))) + 12w(x(y(xz))) + 12x(w(y(xz))) + 6w(z(yx^2)) + 12x(z(y(wx))) + 12w(x(y(xz))) + 12x(w(y(xz))) + 12x(x(y(wz))) - 3z(w(x(xy))) - 3z(x(w(xy))) - 3z(x(x(wy))) - 3w(z(x(xy))) - 3x(z(w(xy))) - 3x(z(x(wy))) - 3w(x(z(xy))) - 3x(w(z(xy))) - 3x(x(z(wy))) - 3w(x(x(zy))) - 3x(w(x(zy))) = 0. (1.15)$$

Let's take  $x=e, y \in A_\lambda, z \in A_\mu$  and  $w \in B$  in (1.15). We have

$$(3 - 6\mu)e(y(wz)) + (3 - 6\lambda)e(z(yw)) + (-3\lambda - 3 + 12\mu - 3\lambda \delta_{\lambda,\mu})e(w(yz)) + (2\mu^2 + \mu - \frac{2}{3}y(wz) + (-3\lambda^2 + 6\lambda - \frac{7}{3})z(yw) + (-8\mu^2 - 4\mu - 3\lambda^2 + 6\lambda + 1 + 6\mu^2 \delta_{\lambda,\mu})w(yz) = 0$$
 (1.16)

with  $\delta_{\lambda,\mu}=1$  if  $\lambda=\mu$  and 0 if not.

For  $\lambda=\mu$  in (1.16), we have

$$(3 - 6\mu)e(y(wz)) + (3 - 6\mu)e(z(yw)) + (-3\lambda - 3 + 12\mu - 3\lambda \delta_{\lambda,\mu})e(w(yz)) + (2\mu^2 + \mu - \frac{2}{3}y(wz) + (-3\lambda^2 + 6\lambda - \frac{7}{3})z(yw) + (-8\mu^2 - 4\mu - 3\lambda^2 + 6\lambda + 1 + 6\mu^2 \delta_{\lambda,\mu})w(yz) = 0.$$
 (1.17)

In (1.17), let's switch  $y$  and  $z$ . It gives

$$(3 - 6\mu)e(y(wz)) + (3 - 6\mu)e(z(yw)) + (6\mu - 3)e(w(yz)) + (2\mu^2 + \mu - \frac{2}{3}y(wz) + (-3\mu^2 + 6\mu - \frac{7}{3})z(yw) + (-5\mu^2 + 2\mu + 1)w(yz) = 0.$$
 (1.18)

Let's subtract (1.17) and (1.18). We have

$$(5\mu^2 - 5\mu + \frac{5}{3})[y(wz) - z(yw)] = 0,$$
 that means  $y(wz) - z(yw) = 0$ . (1.19)

Let's take  $y = y_\lambda, z = z_\lambda$  and  $w = w_2$  in (1.19), we have (i)

Using (1.19) in (1.17), it follows that

$$(6 - 12\mu)e(y(wz)) + (-\mu^2 + 7\mu - 3)y(wz) + (6\mu - 3)e(w(yz)) + (-5\mu^2 + 2\mu + 1)w(yz) = 0. (1.20)$$

If  $\lambda=\mu=0$  in (1.20), we have  $6e(y(wz)) - 3y(wz) - 3e(w(yz)) + w(yz) = 0$ . Let's multiply this last equation by  $e$ , since  $y(wz), w(yz) \in B = Ker(3L_e - 3L_e + I)$ , by simplifying, we can write  $3e(y(wz)) - 2y(wz) - 2e(w(yz)) + w(yz) = 0$ . By comparing the two last equations, it follows that.

$y(wz) + e(w(yz)) - w(yz) = 0$ , which means  $(L_e - I)(w(yz)) = -y(wz)$ . Setting  $y = y_0, z = z_0$  and  $w = w_2$ , we have result (iv).

If we take  $\lambda=\mu=1$  in (1.20), we have  $-6e(y(wz)) + 3y(wz) + 3e(w(yz)) - 2w(yz) = 0$ . Multiplying this equation by  $e$ , since  $y(wz), w(yz) \in B$ , after simplification, we have  $-3e(y(wz)) + 2y(wz) + e(w(yz)) - w(yz) = 0$ . Comparing these two last equations, we have  $-y(wz) + e(w(yz)) = 0$ , which means  $e(w(yz)) = y(wz)$ . Using notations  $y = y_1, z = z_1$  and  $w = w_2$ , we have result (v).

Let us now consider  $\lambda = 0$  and  $\mu = 1$  in (1.16). We have  $(3L_e - \frac{7}{3}I)[z(yw) - y(zw)] = 0$ . Since the only eigen values are 0 and 1, the operator  $3L_e - \frac{7}{3}I$  is injective. So, we can write  $y(zw) = z(yw)$ . Taking  $y = y_1, z = z_0$  and  $w = w_2$  we have  $y_1(z_0w_2) = z_0(y_1w_2)$ .

In the same way, if  $\lambda=1$  and  $\mu=0$  in (1.16), we have  $(3L_e - \frac{2}{3}I)[y(zw) - z(yw)] = 0$ . That means  $y(zw) = z(yw)$ , because the operator  $3L_e - \frac{2}{3}I$  is injective. Using the following notations  $y = y_1, z = z_0$  and  $w = w_2$ , we have  $y_1(z_0w_2) = z_0(y_1w_2)$ .

**Example**

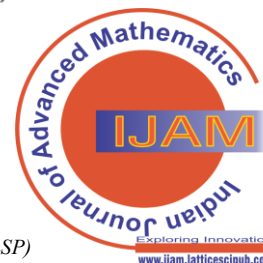
Let's consider the five-dimensional commutative  $\mathbb{R}$ -algebra  $A$ , in which the non-zero products in the basis  $\{e, e_0, e_1, e_2, e_3\}$  are given by:  $e^2 = e, ee_1 = e_1, ee_2 = \frac{1}{2}e_2 - \frac{\sqrt{3}}{6}e_3, ee_3 = \frac{\sqrt{3}}{6}e_2 + \frac{1}{2}e_3$ .  $A$  satisfies (2), with  $A_0 = \langle e_0 \rangle, A_1 = \langle e, e_1 \rangle$  and  $B = \langle e_2, e_3 \rangle$ . In fact, to show that this algebra satisfies (2), we just need to show that it satisfies the total linearization of (2). In this total linearization, if four or five variables are replaced by  $e$ , then equation (2) is satisfied throughout (1.1). All other combinations show that each term cancels.

**B. Polynomial  $3t^2 - 3t + 1$  admits two roots  $\lambda_1$  and  $\lambda_2$  in  $F$**

Let's consider  $p(t) = -t(t - 1)(t - \lambda_1)(t - \lambda_2)$ , in this case,  $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$ , with  $A_\mu = \{x \in A | ex = \mu x\}$ . The following theorem gives relations between the Peirce components.

**Theorem**

Let  $A$  be an algebra defined by (2). Assume that  $3t^2 - 3t + 1 = 3(t - \lambda_1)(t - \lambda_2)$  in  $F[t]$ . Then the Peirce decomposition of  $A$  is given by  $A = A_0 \oplus A_1 \oplus A_{\lambda_1} \oplus A_{\lambda_2}$ , with  $A_0$  and  $A_1$  orthogonal subalgebras,  $A_{\lambda_i}$  a zero algebra,  $A_0A_{\lambda_i} \subseteq A_{\lambda_i}, A_1A_{\lambda_i} \subseteq A_{\lambda_i}$  ( $i=1,2$ ) and



$A_{\lambda_1}A_{\lambda_2} = 0$ . Moreover, we have:

- i.  $z_0(w_0y_{\lambda_i}) + w_0(z_0y_{\lambda_i}) = 2\bar{\lambda}_iy_{\lambda_i}(w_0z_0)$
- ii.  $z_1(w_1y_{\lambda_i}) + w_1(z_1y_{\lambda_i}) = (1 - 3(\bar{\lambda}_i)^3)y_{\lambda_i}(w_1z_1)$
- iii.  $y_1(w_{\lambda_i}z_0) = z_0(y_1w_{\lambda_i})$

with  $\bar{\lambda}_i$  the conjugate number of  $\lambda_i$ , and  $y_r, z_s, w_t$  are respectively elements of  $A_r, A_s$  and  $A_t$ .

Proof.

Then taking  $x=e, y \in A_{\bar{\lambda}}$  and  $z \in A_{\lambda}$  in (1.2) we have

$$[-3L_e^3 + (12\lambda - 3\bar{\lambda})L_e^2 + (-4\lambda - 8\lambda^2 + 6\bar{\lambda} - 3\bar{\lambda}^2)L_e + (\lambda + \lambda^2 + 2\lambda^3 - 4\bar{\lambda} + 6\bar{\lambda}^2 - 3\bar{\lambda}^3)I](yz) = 0. \quad (1.20)$$

Let's consider  $q_{\bar{\lambda}\{\lambda\}}(t) = -3t^3 + (12\lambda - 3\bar{\lambda})t^2 + (-8\lambda^2 + 6\bar{\lambda} - 3\bar{\lambda}^2 - 4\lambda)t + \lambda + \lambda^2 + 2\lambda^3 - 4\bar{\lambda} + 6\bar{\lambda}^2 - 3\bar{\lambda}^3$  have

Taking  $\bar{\lambda} = \lambda = 1$ , we have  $q_{1,1}(t) = -3t^3 + 9t^2 - 9t + 3 = -3(t-1)^3$  and  $gcd(p(t), q_{1,1}(t)) = t - 1$ .

Then, it gives  $A_1A_1 \subseteq A_1$ .

If  $\bar{\lambda} = \lambda = 0$ , then  $q_{0,0}(t) = -3t^3$  and  $gcd(p(t), q_{0,0}(t)) = t$ , and that leads to  $A_0A_0 \subseteq A_0$ . For  $\bar{\lambda} = 1$  and

$\lambda = 0$ , we have  $q_{1,0}(t) = -3t^3 - 3t^2 + 3t - 1$ . Since  $q_{1,0}(\lambda_i) = -2\lambda_i + 1 \neq 0$  ( $i = 1, 2$ ) and 0, 1 are not roots of  $q_{1,0}(t)$ , then  $gcd(p(t), q_{1,0}(t)) = 1$ , and finally  $A_0A_1 = 0$ . Now, since  $\lambda_i$  ( $i = 1, 2$ ) is the root of  $3t^2 - 3t + 1$ , then we have  $\lambda_1 + \lambda_2 = 1, \lambda_1\lambda_2 = \frac{1}{3}$ ,  $3\lambda_i^2 = 3\lambda_i - 1$  and  $3\lambda_i^3 = 2\lambda_i - 1$ .

We shall use these equations to calculate  $q_{\bar{\lambda}\lambda}(t)$  when  $\bar{\lambda}$  or  $\lambda$  equals  $\lambda_i$  ( $i=1,2$ ). Then, for  $\bar{\lambda} = 0$  and  $\lambda = \lambda_i$ , we have

$q_{0,\lambda_i} = -3t^3 + 12\lambda_it^2 - (12\lambda_i - \frac{8}{3})t + \frac{10}{3}\lambda_i - 1$  which roots are different from 0, 1, and  $\lambda_j$  ( $j \neq i$ ). On the other hand, we have  $q_{0,\lambda_i}(\lambda_i) = 0$ , which leads to  $gcd(p(t), q_{0,\lambda_i}(t)) = t - \lambda_i$ , and

$A_0A_{\lambda_i} \subseteq A_{\lambda_i}$ . Now let's consider  $\bar{\lambda} = 1$  and  $\lambda = \lambda_i$ , we have

$q_{1,\lambda_i}(t) = -3t^3 + (12\lambda_i - 3)t^2 + (-12\lambda_i + \frac{17}{3})t + \frac{10}{3}\lambda_i - 2$  which roots are different from 0, 1, and  $\lambda_j$  ( $j \neq i$ ). However, we have,  $q_{1,\lambda_i}(\lambda_i) = 0$ , that means  $gcd(p(t), q_{1,\lambda_i}(t)) = t - \lambda_i$ , and

$A_1A_{\lambda_i} \subseteq A_{\lambda_i}$ . Setting  $\bar{\lambda} = \lambda = \lambda_i$ , we can write  $q_{\lambda_i,\lambda_i}(t) = -3t^3 + 9\lambda_it^2 + (-9\lambda_i + \frac{11}{3})t + \frac{10}{3}\lambda_i - 2$ . The roots of this last polynomial are different from 0, 1,  $\lambda_1$  and  $\lambda_2$ . Then, we have  $gcd(p(t), q_{\lambda_i,\lambda_i}(t)) = 1$  and then  $A_{\lambda_i}A_{\lambda_i} = 0$ .

Let's consider  $\bar{\lambda} = \lambda_i$  and  $\lambda = \lambda_j$  ( $i \neq j$ ). We can write

$q_{\lambda_i,\lambda_j}(t) = -3t^3 + (12\lambda_j - 3\lambda_i)t^2 + (-12\lambda_j + 3\lambda_i + \frac{11}{3}t + \frac{10}{3}\lambda_j - 2$

which roots are different from 0, 1,  $\lambda_1$  and  $\lambda_2$ . Then we have  $gcd(p(t), q_{\lambda_i,\lambda_j}(t)) = 1$ , that means  $A_{\lambda_i}A_{\lambda_j} = 0$  ( $i \neq j$ ).

Partially linearizing (1.2) gives (1.15).

Taking  $x=e, y \in A_{\bar{\lambda}}, z \in A_{\lambda}$  and  $w \in A_{\mu}$  in equation (1.15), we have

$$\begin{aligned} &\mu y(wz) + 2\mu^2 y(wz) + \lambda y(wz) + 2\mu y(e(wz)) + \\ &2\lambda^2 y(wz) + 2\lambda y(e(wz)) + 2y(e(e(wz))) - \\ &4\mu z(yw) - 8\mu^2 z(yw) - 4\lambda w(yz) - 8\mu e(y(wz)) - \\ &8\lambda^2 w(yz) - 8\lambda e(y(wz)) - 8e(y(e(wz))) + \\ &6\bar{\lambda} z(yw) + 12\mu z(e(yw)) + 6\bar{\lambda} w(yz) + 12\mu e(z(yw)) + \\ &12\lambda w(e(yz)) + 12\lambda e(w(yz)) + 12e(e(y(wz))) - \\ &3\bar{\lambda}^2 z(yw) - 3\bar{\lambda} z(e(yw)) - 3z(e(e(yw))) - 3\bar{\lambda}^2 w(yz) - \\ &3\bar{\lambda} e(z(yw)) - 3e(z(e(yw))) - 3\bar{\lambda} w(e(yz)) - \\ &3\bar{\lambda} e(w(yz)) - 3e(e(z(yw))) - 3w(e(e(yz))) - \\ &3e(w(e(yz))) - 3e(e(w(yz))) = 0. \end{aligned}$$

On the other hand, if we take  $\lambda = \mu = 0$  and  $\bar{\lambda} = \lambda_i$ , we have

$(3\lambda_i - 1)z(yw) + (3\lambda_i - 1)w(yz) = 2\lambda_i y(wz)$ . Since  $\lambda_i$  is a root of  $3t^2 - 3t + 1$ ,

then we have  $3\lambda_i^2 = 3\lambda_i - 1, \lambda_i i\bar{\lambda}_i = \frac{1}{3}$ , and the last equation gives

$z(yw) + w(yz) = 2\bar{\lambda}_i y(wz)$ . Let's take  $\lambda = \mu = 1$  and  $\bar{\lambda} = \lambda_i$ , it gives

$z(yw) + w(yz) = (1 - 3\bar{\lambda}_i^3)y(wz)$ . Considering  $\bar{\lambda} = 1, \lambda = 0$  and  $\mu = \lambda_i$ ,

we have  $y(wz) = z(yw)$ . Let's take  $y = y_{\lambda}, z = z_{\lambda}$  and  $w = w_{\mu}$  in each case, we have the results of the theorem.  $\square$

#### Example

Let A be the five-dimensional algebra, whose non zero products in the basics  $\{e_1, e_2, e_3, e_4, e_5\}$  are given by

$e_1^2 = e_2, e_1e_2 = e_3, e_2^2 = e_4, e_1e_3 = e_5$ . This algebra satisfies (2). In fact,

$$A^2 = \langle e_2, e_3, e_4, e_5 \rangle, \quad A^3 = \langle e_3, e_4, e_5 \rangle, \quad A^4 = \langle e_5 \rangle \text{ and } A^5 = 0.$$

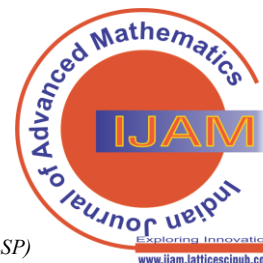
#### Example

Let A be the five-dimensional algebra, whose non zero products in the basics  $\{e, x_0, y_1, z_{\lambda}, w_{\bar{\lambda}}\}$  are given by

$e^2 = e, ey_1 = y_1, ez_{\lambda} = \lambda z_{\lambda}, ew_{\bar{\lambda}} = \bar{\lambda} w_{\bar{\lambda}}, x_0z_{\lambda} = z_{\lambda}, y_1w_{\bar{\lambda}} = w_{\bar{\lambda}}$ . To show that A satisfies (2), we need to show that, regardless of how we replace the variables by elements of the basis, the complete linearization of (2) is satisfied. If four or five variables are replaced by e, equation (2) is confident throughout (1.1). If three variables are replaced by e, and the two others chosen between  $w_{\bar{\lambda}}$ , equation (2) is satisfied throughout (1.21). If two variables are replaced by e, each term cancels. Regardless of the combination of elements in the basis that is used in the linearization of (2), each term cancels, and (2) is trivially satisfied. Finally, A satisfies identity (2).

**Note:** In the previous theorem, the subspaces  $J_0 = \{x \in A_0 / x(A_{\bar{\lambda}} + A_{\bar{\lambda}}) = 0\}$ ,

$J_0 = \{x \in A_0 / x(A_{\bar{\lambda}} + A_{\bar{\lambda}}) = 0\}$ , and  $J = J_0 + J_1$  are ideals of A. In fact, let's consider  $x \in J_0$  and we can write  $b \in A_{\lambda} + A_{\bar{\lambda}}$ , we have



$(xa)b = (xa_0)b_\lambda + (xa_0)b_{\bar{\lambda}} = 0$  by (i). Then  $J_0$  is an ideal. Similarly, we demonstrate that it is an ideal of  $A$ . Moreover, for all  $p, q \in A_0$ , we have  $z = (p^2q)p - p^2(qp) \in J_0$ . In fact, let's consider the F-linear map  $(i=1,2)$ , with  $\mu = 2\bar{\lambda}_i$ . Since  $x_0(y_0z_{\lambda_i} + y_0(x_0z_{\lambda_i})) = \mu(x_0y_0)z_{\lambda_i}$ , that means  $S_{x_0}S_{y_0} + S_{y_0}S_{x_0} = S_{x_0y_0}$ , the map  $S_i : A_0 \rightarrow \text{End}_F(A_{\lambda_i})$ ,  $x_0 \mapsto 2S_{x_0}$  satisfies being the product in  $\text{End}_F(A_{\lambda_i})^+$ . Then  $S_i$  is a morphism of algebras. Its kernel  $J_{0i} = \text{Ker}S_i = \{x_0 \in A_0 / A_{\lambda_i} = 0\}$  is an ideal of  $A_0$ . We can easily see that it is an ideal as an intersection of ideals. Since the kernel of the algebra morphism of  $A_0$  into the special Jordan algebra  $\text{End}(A_{\lambda_i})^+$ , then  $z = (p^2q)p - p^2(qp)$  is an element of  $J_{0i}$  ( $i=1,2$ ), that means  $z$  is also an element of  $J_0$ .

### III. IDENTITY (3)

Now, we study the algebra defined by identity (3). Let's take  $x=e$  in (3), we have  $(4L_e^4 - 8L_e^3 + 5L_e^2 - L_e)(y) = 0$ . Let's consider  $p(t) = 4t^4 - 8t^3 + 5t^2 - t = t(t-1)(2t-1)^2$ .

Then, the Peirce decomposition of  $A$  is  $A = A_1 \oplus B_2 \oplus A_0$ , with  $B_2 = \{x \in A / (2L_e - I)^2(x) = 0\}$ . Define the following subspaces:  $C_{\frac{1}{2}} = \{y / \exists x \in B_2, y = (2L_e - I)(x)\}$  and  $A_{\frac{1}{2}} = \{x \in A / ex = \frac{1}{2}x\}$ . We have  $C_{\frac{1}{2}} \subseteq A_{\frac{1}{2}} \subseteq B_2$ .

#### Theorem

Let  $A$  be an algebra satisfying (3). Then we have  $A = A_1 \oplus B_2 \oplus A_0$ , with  $A_0$  et  $A_1$  being orthogonal subspaces,  $A_0B_2 \subseteq B_{\frac{1}{2}}$ ,  $A_1B_2 \subseteq B_2$ ,  $A_0A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$ ,  $A_1A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$ ,  $A_0C_{\frac{1}{2}} \subseteq C_{\frac{1}{2}}$ ,  $A_1C_{\frac{1}{2}} \subseteq C_{\frac{1}{2}}$ ,  $A_{\frac{1}{2}}C_{\frac{1}{2}} = 0$ ,  $B_2C_{\frac{1}{2}} \subseteq C_{\frac{1}{2}}$ ,  $A_{\frac{1}{2}} \subseteq A_0 + A_{\frac{1}{2}} + A_1$ ,  $A_{\frac{1}{2}}B_2 \subseteq A_0 + A_{\frac{1}{2}} + A_1$ ,  $B_2B_2 \subseteq A_0 + B_2 + A_1$ .

Proof.

By partially linearizing (3), we have

$$-4y((xz)x^2 + 5z(yx^3) + 5x(y(zx^2)) + 10x(y(x(xz)))) - 9z(x(yx^2)) - 9x(z(yx^2)) - 18x(x(y(yxz))) + 4z(x(x(xy))) + 4x(z(x(xy))) + 4x(x(z(xy))) + 4x(x(x(zy))) + z((yx)x^2) + x((yz)x^2) + 2x((xy)(xz)) + 2(xz)(yx^2) + 2x^2(y(xz)) - (yz)x^3 - (yx)(zx^2) - 2(yx)(x(xz)) = 0. \quad (2.1)$$

Taking  $x=e$  in equation (2.1), we have

$$-4y(e(ez)) + 5z(ey) + 5e(y(ez)) + 10e(y(e(ez))) - 8z(e(ey)) - 9e(z(ey)) - 18e(e(y(ez))) + 4z(e(e(ey))) + 4e(z(e(ey))) + 4e(e(z(ey))) + 4e(e(e(zy))) + e(e(yz)) + 2e((ey)(ez)) + (ez)(ye) + 2e(y(ez)) - e(yz) - 2(ey)(e(ez)) = 0, that gives  $(4L_e^3 + L_e^2 - L_e)(yz) + L_e^2[-18y(ez) + 4z(ey)] + L_e[5y(ez) + 10y(e(ez)) - 9z(ey) + 4z(e(ey))] + 2(ey)(ez) + 2y(ez) - 4y(e(ez)) + z[4L_e^3 - 8L_e^2 + 5L_e](y) + (ey)(ez) - 2(ey)(e(ez)) = 0. \quad (2.2)$$$

Now take  $y \in A_\lambda$  and  $z \in A_\mu$  in (2.2), we can write

$$[4L_e^3 + (1 - 18\mu + 4\lambda)L_e^2 + (-1 + 7\mu + 10\mu^2 - 9\lambda + 4\lambda^2 + 2\lambda\mu)L_e + (-4\mu^2 + \lambda\mu - \lambda\mu^2 + 4\lambda^3 - 8\lambda^2 + 5\lambda\mu)](yz) = 0.$$

Let's set

$$q_{\lambda,\mu}(t) = 4t^3 + (1 - 18\mu + 4\lambda)t^2 + (-1 + 7\mu + 10\mu^2 - 9\lambda + 4\lambda^2 + 2\lambda\mu)t - 4\mu^2 + \mu - \lambda\mu^2 + 4\lambda^3 - 8\lambda^2 + 5\lambda.$$

For  $\lambda = \mu = 1$ , we have  $q_{1,1}(t) = 4t^3 - 13t^2 + 13t - 4$ . That gives  $\text{gcd}(p(t), q_{1,1}(t)) = t - 1$ , otherwise  $A_1A_1 \subseteq A_1$ .

For  $\lambda = 1$  and  $\mu = 0$ , we have

$$q_{1,0}(t) = 4t^3 + 5t^2 - 6t + 1 \text{ and } \text{gcd}(p(t), q_{1,0}(t)) = 1, \text{ that means, } A_0A_1 = 0. \text{ If } \lambda = \mu = 0, \text{ we have } q_{0,0}(t) = 4t^3 + t^2 - t \text{ and } \text{gcd}(p(t), q_{0,0}(t)) = t, \text{ this leads to } A_0A_0 \subseteq A_0.$$

Let's consider  $u = ey - \frac{1}{2}y$ , with  $y \in B_2$ . Then we have  $eu = \frac{1}{2}u$ ,  $ey = \frac{1}{2}y + u$ ,  $e(ey) = \frac{1}{4}y + u$ ,  $e(e(ey)) = \frac{1}{8}y + \frac{3}{4}u$ . Taking  $z \in A_\lambda$ , equation (2.2) becomes

$$[4L_e^3 + (3 - 18\lambda)L_e^2 + (-\frac{9}{2} + 8\lambda + 10\lambda^2)L_e + 1 + \frac{1}{2}\lambda - 5\lambda^2](yz) + [4L_e^2 + (2\lambda - 5)L_e + (-2\lambda^2 + \lambda)]I(zu) = 0.$$

If  $\lambda = 0$  and  $u = 0$ , that means  $z \in A_0$  and  $y \in A_{\frac{1}{2}}$ , then the last equation can be written

$$(4L_e^3 + 3L_e^2 - \frac{9}{2}L_e + 1)(yz) = 0. \text{ Setting } q(t) = 4t^3 + 3t^2 - \frac{9}{2}t + 1 = \frac{1}{2}(2t-1)(4t^2 + 5t - 2), \text{ we have } \text{gcd}(p(t), q(t)) = 2t - 1 \text{ which means } yz \in A_{\frac{1}{2}} \text{ then } A_0A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}.$$

If  $\lambda = 0$  and  $u \neq 0$  which means  $z \in A_0$  and  $y \in B_2$ , then we have  $(4L_e^3 + 3L_e^2 - \frac{9}{2}L_e + I)(yz) + (4L_e^2 - 5L_e)(zu) = 0$ . Since  $u \in C_{\frac{1}{2}}$  and  $z \in A_{\frac{1}{2}}$ , the last equation becomes

$$\frac{1}{2}(2L_e - 1)(4L_e^2 + 5L_e - 2I)(yz) = \frac{3}{2}zu. \text{ Then we have } A_0B_2 \subseteq B_2.$$

If  $\lambda = 1$  and  $u = 0$ , that means,  $z \in A_1$  and  $y \in A_{\frac{1}{2}}$ , and we can write

$$(4L_e^3 - 15L_e^2 + \frac{27}{2}L_e - \frac{7}{2}I)(yz) = 0.$$

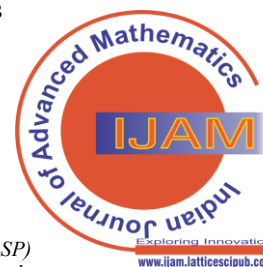
Let's set

$$q(t) = 4t^3 - 15t^2 + \frac{27}{2}t - \frac{7}{2} = \frac{1}{2}(2t-1)(4t^2 - 13t + 7)$$

we have  $\text{gcd}(p(t), q(t)) = 2t - 1$ .

Then  $yz \in A_{\frac{1}{2}}$  gives  $A_1A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$ .

If  $\lambda=1$  and  $u \neq 0$ , that means  $z \in A_1$  and  $y \in B_2$ , then



## Study of Some Degree Five Identities of Type (4,1)

we have

$$(4L_e^3 - 15L_e^2 + \frac{27}{2}L_e - \frac{7}{2}I)(yz) + (4L_e^2 - 3L_e - 1)(zu) = 0$$

. Since  $u \in C_{\frac{1}{2}}$  and  $zu \in A_{\frac{1}{2}}$ , this last equation gives

$$\frac{1}{2}(2L_e - 1)(4L_e^2 - 13L_e + 7I)(yz) = \frac{3}{2}zu, \text{ which means}$$

$yz \in B_2$ , and finally

$$A_1 B_2 \subseteq B_2.$$

Now let's take  $y, z \in B_2$  and set  $u = ey - \frac{1}{2}y$  and

$$v = ez - \frac{1}{2}z. \text{ We can write } eu = \frac{1}{2}u, ey = \frac{1}{2}y + u,$$

$$e(ey) = \frac{1}{4}y + u, e(e(ey)) = \frac{1}{8}y + \frac{3}{4}u. \text{ In the same way,}$$

$$\text{we have } ev = \frac{1}{2}v, ez = \frac{1}{2}z + v,$$

$$e(ez) = \frac{1}{4}z + v, e(e(ez)) = \frac{1}{8}z + \frac{3}{4}v. \text{ Using this in}$$

(2.2), we have

$$(4L_e^3 - 6L_e + 2L_e)(yz) + (-18L_e^2 + 18L_e - \frac{9}{2}I)(yv) + (4L_e^2 - 4L_e)(zu) + (2L_e - I)(uv) = 0,$$

which means

$$2L_e(2L_e - I)(L_e - I)(yz) - \frac{9}{2}(2L_e - I)^2(yv) + 4L_e(L_e - I)(zu) + (2L_e - I)(uv) = 0. \text{ (2.3)}$$

In the following part of the proof, we shall note  $x_\lambda$  ( $i \in \{0, \frac{1}{2}, 1\}$ ) the component of an element  $x$  in  $A$  with regard to subspaces  $A_\lambda$  and  $x_2$  the component in  $B_2$ .

If  $u = v = 0$ , that means  $y, z \in A_{\frac{1}{2}}$ , then (2.3) becomes

$$2L_e(2L_e - I)(L_e - I)(yz) = 0. \text{ Then we have } yz \in A_0 + A_{\frac{1}{2}} + A_1, \text{ which means } A_{\frac{1}{2}}A_{\frac{1}{2}} \subseteq A_0 + A_{\frac{1}{2}} + A_1.$$

If  $u = 0$  and  $v \neq 0$ , that means  $y \in A_{\frac{1}{2}}$  and  $z \in B_2$ , then (2.3) gives

$$2L_e(L_e - I)(2L_e - I)(yz) - \frac{9}{2}(2L_e - I)^2(yv) = 0. \text{ Let's set } yz = (yz)_0 + (yz)_2 + (yz)_1. \text{ Since } yv \in A_0 + A_{\frac{1}{2}} + A_1,$$

taking  $yv = (yv)_0 + (yv)_{\frac{1}{2}} + (yv)_1$  the last equation

$$\text{gives } \frac{1}{2}(2L_e - I)((yz)_2) - \frac{9}{2}[(yv)_0 + (yv)_1] = 0, \text{ because } (2L_e - I)((yz)_2) \in C_{\frac{1}{2}}. \text{ Then we have}$$

$$(2L_e - I)((yz)_2) = (yv)_0 = (yv)_1 = 0. \text{ Because of, } (yv)_0 = (yv)_1 = 0, \text{ we can write } yv \in A_{\frac{1}{2}}, \text{ that is,}$$

$$A_{\frac{1}{2}}C_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}. \text{ From } (2L_e - I)((yz)_2) = 0, \text{ we can write } (yz)_2 = (yz)_{\frac{1}{2}}, \text{ which means, } A_{\frac{1}{2}}B_2 \subseteq A_0 + A_{\frac{1}{2}} + A_1.$$

On the other hand, taking  $v = 0$  and  $u \neq 0$  in (2.3), we have  $2L_e(2L_e - I)(L_e - I)(yz) - zu = 0$ . Since  $yz \in A_0 + A_{\frac{1}{2}} + A_1$  we can write  $zu = 0$ . Then, we have

$$A_{\frac{1}{2}}C_{\frac{1}{2}} = 0.$$

If  $u \neq 0$  and  $v \neq 0$ , (2.3) can be written

$$\frac{1}{2}(2L_e - I)(yz)_2 - \frac{9}{2}[(yv)_0 + (yv)_1] - (zu)_{\frac{1}{2}} = 0.$$

Otherwise  $\frac{1}{2}(2L_e - I)((yz)_2) = (zu)_{\frac{1}{2}}$  and

$(yv)_0 = (yv)_1 = 0$ . Equation  $(yv)_0 = (yv)_1 = 0$  leads to  $yv \in A_{\frac{1}{2}}$  that means  $C_{\frac{1}{2}}B_2 \subseteq A_{\frac{1}{2}}$ . But

$\frac{1}{2}(2L_e - I)(yz)_2 = (zu)_{\frac{1}{2}}$  gives  $(zu)_{\frac{1}{2}} \in C_{\frac{1}{2}}$  and finally, we have

$$C_{\frac{1}{2}}B_2 \subseteq C_{\frac{1}{2}}. \square$$

Products of three factors are given in the following proposition.

### Proposition

Let  $A$  be an algebra defined by identity (3). Then we have

- i.  $(z_0 y_0) x_{\frac{1}{2}} = z_0 (y_0 x_{\frac{1}{2}}) + y_0 (z_0 x_{\frac{1}{2}})$
- ii.  $(z_1 y_1) x_{\frac{1}{2}} = z_1 (y_1 x_{\frac{1}{2}}) + y_1 (z_1 x_{\frac{1}{2}})$
- iii.  $\left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_0 \right) \right]_1 = \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_0 \right) \right]_1$
- iv.  $\left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_0 \right) \right]_1 = \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_0 \right) \right]_1$
- v.  $\left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_0 \right) \right]_{\frac{1}{2}} = \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_0 \right) \right]_{\frac{1}{2}}$
- vi.  $z_0 \left( x_{\frac{1}{2}} y_{\frac{1}{2}} \right)_0 = \left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_0 \right) + y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_0 \right) \right]_0$
- vii.  $\left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_1 \right) \right]_1 = \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_1 \right) \right]_1$
- viii.  $\left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_1 \right) \right]_{\frac{1}{2}} = \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_1 \right) \right]_{\frac{1}{2}}$
- ix.  $z_1 \left( x_{\frac{1}{2}} y_{\frac{1}{2}} \right)_1 = 2 \left[ x_{\frac{1}{2}} \left( y_{\frac{1}{2}} z_1 \right) \right]_1 = 2 \left[ y_{\frac{1}{2}} \left( x_{\frac{1}{2}} z_1 \right) \right]_1$ .

Proof.

Partially linearizing (2.1), it gives

$$\begin{aligned} & -4y((wz)x^2) - 8y((xw)(xz)) + 5z(y(wx^2)) + 10z(y(xxw)) \\ & + 5w(y(zx^2)) + 10x(y(zxw)) + 10w(y(xxz)) + 10x(y(wxz)) \\ & + 10x(y(xwz)) - 9z(w(yx^2)) - 18z(x(yxw)) - 9w(z(yx^2)) \\ & - 18x(z(yxw)) - 18w(x(yxz)) - 18x(w(yxz)) - \\ & 18x(x(ywz)) + 4z(w(xxy)) + 4z(x(wxy)) + \\ & 4z(x(xwy)) + 4w(z(xxy)) + 4x(z(wxy)) + \\ & 4x(z(xyw)) + 4w(x(zxy)) + 4x(w(zxy)) + \\ & 4x(x(zwy)) + 4w(x(xzy)) + 4x(w(xzy)) + \\ & 4x(x(wzy)) + z((wy)x^2) + 2z((xy)(xw)) + w((yz)x^2) + \\ & 2x((yz)(xw)) + 2w((xy)(xz)) + 2x((wy)(xz)) + 2x((xy)(wz)) + \\ & 2(wz)(yx^2) + 4(xz)(ywx) + 4(xw)(y(xz)) + 2x^2(ywz) - \\ & (yz)(wx^2) - 2(yz)(x(xw)) - (yw)(zx^2) - 2(yx)(z(xw)) - \end{aligned}$$



$$2(yw)(x(xz)) - 2(yx)(w(xz)) - 2(yx)(x(wz)) = 0. \tag{3.4}$$

Let's consider  $x=e$ ,  $y \in A_\gamma$ ,  $z \in A_\lambda$  and  $w \in A_{\frac{\lambda}{2}}$  (3.4), we have

$$\begin{aligned} & -2y(wz) - 4\lambda y(wz) + \frac{5}{2}z(wy) + \frac{5}{2}z(wy) + 5\lambda\delta_{\gamma,\lambda}w(yz) + \\ & \frac{5}{2}y(wz) + 10\lambda^2w(yz) + 5\lambda y(wz) + \frac{5}{2}y(wz) - 9\gamma z(wy) \\ & - \frac{9}{2}z(wy) - 9\gamma\delta_{\gamma,\lambda}w(yz) - \frac{9}{2}z(wy) - 18\lambda^2\delta_{\gamma,\lambda}w(yz) - \\ & 9\lambda\delta_{\gamma,\lambda}w(yz) - \frac{9}{2}y(wz) + 4\gamma^2 z(wy) + 2\gamma z(wy) + \\ & z(wy) + 4\gamma^2\delta_{\gamma,\lambda}w(yz) + 2\gamma z(wy) + z(wy) + \\ & 4\gamma\delta_{\gamma,\lambda}w(yz) + 2\gamma\delta_{\gamma,\lambda}w(yz) + z(wy) + 4\gamma^2\delta_{\gamma,\lambda}w(yz) + \\ & 2\gamma\delta_{\gamma,\lambda}w(yz) + \delta_{\gamma,\lambda}w(yz) + \frac{1}{2}z(wy) + \gamma z(wy) + \gamma\delta_{\gamma,\lambda}w(yz) + \\ & \frac{1}{2}\delta_{\gamma,\lambda}w(yz) + 2\gamma\lambda\delta_{\gamma,\lambda}w(yz) + \lambda z(wy) + \gamma y(wz) + \\ & 2\gamma y(wz) + 2\lambda z(wy) + y(wz) + 2\lambda\delta_{\gamma,\lambda}w(yz) - \\ & \frac{1}{2}\delta_{\gamma,\lambda}w(yz) - \frac{1}{2}\delta_{\gamma,\lambda}w(yz) - \lambda z(wy) - \gamma y(wz) - \\ & 2\lambda^2 z(wy) - 2\gamma\lambda y(wz) - \gamma y(wz) = 0, \end{aligned}$$

that gives after simplifying

$$\begin{aligned} & \left(-\frac{1}{2} + \lambda - 2\gamma\lambda + \gamma\right)y(wz) + \\ & \left(-\frac{1}{2} + 2\lambda - 2\lambda^2 - 4\gamma + 4\gamma^2\right)z(wy) + \\ & (10\lambda^2 - 10\gamma\delta_{\gamma,\lambda} + \frac{1}{2}\delta_{\gamma,\lambda})w(yz), \tag{3.5} \end{aligned}$$

in which  $\delta_{\gamma,\lambda}$  equals 1 if  $\gamma = \lambda$  and 0 if not.

Taking  $\gamma = \lambda = 0$ , in (3.5), we have

$y(zw) + z(yw) = w(yz)$ . In the same way, let's consider  $\gamma = \lambda = 0$  in

(3.5), we have  $y(wz) + z(wy) = w(yz)$ . On the other hand, considering  $\gamma = 1$  et  $\lambda = 0$ , it gives  $y(wz) = z(wy)$ . In each case, let's note  $y = y_\gamma$ ,  $z = z_\lambda$  and  $w = w_{\frac{\lambda}{2}}$ , the result

1) follows.

Now let's consider  $x=e$ ,  $y \in A_\lambda$  with  $\lambda = 0,1$ , and  $w, z \in A_{\frac{\lambda}{2}}$ .

Write  $wz = (wz)_0 + (wz)_{\frac{1}{2}} + (wz)_1$ ,

$w(yz) = [w(yz)]_0 + [w(yz)]_{\frac{1}{2}} + [w(yz)]_1$ ,

$z(wy) = [z(wy)]_0 + [z(wy)]_{\frac{1}{2}} + [z(wy)]_1$ ,

$y(wz) = y(wz)_\lambda + y(wz)_{\frac{\lambda}{2}}$ . Using these equations in (3.4),

we have

$$\begin{aligned} & -2y(wz)_{\frac{1}{2}} - 4\lambda y(wz)_1 - 2y(wz)_\lambda - 2y(wz)_{\frac{1}{2}} + \frac{5}{2}[z(wy)]_0 + \\ & \frac{5}{2}[z(wy)]_{\frac{1}{2}} + \frac{5}{2}[z(wy)]_1 + \frac{5}{2}[z(wy)]_0 + \\ & \frac{5}{2}[z(wy)]_{\frac{1}{2}} + \frac{5}{2}[z(wy)]_1 + \frac{5}{2}[w(yz)]_0 + \\ & \frac{5}{2}[w(yz)]_{\frac{1}{2}} + \frac{5}{2}[w(yz)]_1 + 5\lambda y(wz)_\lambda + \\ & \frac{5}{2}y(wz)_{\frac{1}{2}} + \frac{5}{2}[w(yz)]_0 + \frac{5}{2}[w(yz)]_{\frac{1}{2}} + \\ & \frac{5}{2}[w(yz)]_1 + 5\lambda y(wz)_\lambda + \frac{5}{2}y(wz)_{\frac{1}{2}} + \frac{5}{2}y(wz)_{\frac{1}{2}} + \\ & 10\lambda^2 y(wz)_\lambda - 9\lambda[z(wy)]_0 - 9\lambda[z(wy)]_{\frac{1}{2}} - 9\lambda[z(wy)]_1 - \end{aligned}$$

$$\begin{aligned} & \frac{9}{2}[z(wy)]_0 - \frac{9}{2}[z(wy)]_{\frac{1}{2}} - \frac{9}{2}[z(wy)]_1 \\ & - 9\lambda[w(yz)]_0 - 9\lambda[w(yz)]_{\frac{1}{2}} - 9\lambda[w(yz)]_1 - \frac{9}{2}[z(wy)]_{\frac{1}{2}} - \\ & 9[z(wy)]_1 - \frac{9}{2}[w(yz)]_0 - \frac{9}{2}[w(yz)]_{\frac{1}{2}} - \frac{9}{2}[w(yz)]_1 - \\ & \frac{9}{2}[w(yz)]_{\frac{1}{2}} - 9[w(yz)]_1 - 18\lambda^2 y(wz)_\lambda - \frac{9}{2}y(wz)_{\frac{1}{2}} + \\ & 2[z(wy)]_1 + 2\lambda[w(yz)]_0 + 2\lambda[w(yz)]_{\frac{1}{2}} + 2\lambda[w(yz)]_1 + \\ & 2\lambda[w(yz)]_{\frac{1}{2}} + 4\lambda[w(yz)]_1 + [w(yz)]_{\frac{1}{2}} + \\ & 4[w(yz)]_1 + [w(yz)]_0 + [w(yz)]_{\frac{1}{2}} + [w(yz)]_1 + \\ & [w(yz)]_{\frac{1}{2}} + 2[w(yz)]_1 + 4[w(yz)]_1 + \frac{1}{2}[z(wy)]_0 + \\ & \frac{1}{2}[z(wy)]_{\frac{1}{2}} + \frac{1}{2}[z(wy)]_1 + \lambda[z(wy)]_0 + \lambda[z(wy)]_{\frac{1}{2}} + \\ & \lambda[z(wy)]_1 + \frac{1}{2}[w(yz)]_0 + \frac{1}{2}[w(yz)]_{\frac{1}{2}} + \frac{1}{2}[w(yz)]_1 + \\ & \frac{1}{2}[w(yz)]_{\frac{1}{2}} + [w(yz)]_1 + \lambda[w(yz)]_0 + \lambda[w(yz)]_{\frac{1}{2}} + \\ & \lambda[w(yz)]_1 + \frac{1}{2}[z(wy)]_{\frac{1}{2}} + [z(wy)]_1 + 2\lambda^2 y(wz)_\lambda + \\ & \lambda y(wz)_{\frac{1}{2}} + 2\lambda y(wz)_\lambda + 2\lambda y(wz)_{\frac{1}{2}} + [z(wy)]_0 + \\ & [z(wy)]_{\frac{1}{2}} + [z(wy)]_1 + 2\lambda y(wz)_\lambda + y(wz)_{\frac{1}{2}} + [w(yz)]_0 + \\ & [w(yz)]_{\frac{1}{2}} + [w(yz)]_1 - \frac{1}{2}[w(yz)]_0 - \frac{1}{2}[w(yz)]_{\frac{1}{2}} - \frac{1}{2}[w(yz)]_1 - \\ & \frac{1}{2}[w(yz)]_0 - \frac{1}{2}[w(yz)]_{\frac{1}{2}} - \frac{1}{2}[w(yz)]_1 - \frac{1}{2}[z(wy)]_0 - \\ & \frac{1}{2}[z(wy)]_{\frac{1}{2}} - \frac{1}{2}[z(wy)]_1 - \lambda y(wz)_\lambda - \lambda y(wz)_{\frac{1}{2}} - \\ & \frac{1}{2}[z(wy)]_0 - \frac{1}{2}[z(wy)]_{\frac{1}{2}} - \frac{1}{2}[z(wy)]_1 \\ & - \lambda y(wz)_\lambda - \lambda y(wz)_{\frac{1}{2}} - \lambda y(wz)_{\frac{1}{2}} - 2\lambda y(wz)_1 = 0, \end{aligned}$$

that gives after simplifying

$$\begin{aligned} & -6\lambda y(wz)_1 + (-2 + 12\lambda - 6\lambda^2)y(wz)_\lambda + (2 - 8\lambda + 6\lambda^2)[z(wy)]_0 + \\ & (-1 - 4\lambda + 4\lambda^2)[z(wy)]_{\frac{1}{2}} + (-4 - 2\lambda + 4\lambda^2)[z(wy)]_1 + (2 - 6\lambda + \\ & 4\lambda^2)[w(yz)]_0 + (1 - 4\lambda + 4\lambda^2)[w(yz)]_{\frac{1}{2}} + (4 - 2\lambda + 4\lambda^2)[w(yz)]_1 = 0. \end{aligned}$$

Take  $y \in A_0$ , that means  $\lambda = 0$  in (2.6), we have  $[z(wy)]_1 = [w(yz)]_1$ ,  $[z(wy)]_0 = y(wz)_0$  and  $[w(yz)]_{\frac{1}{2}} = [z(wy)]_{\frac{1}{2}}$ .

If  $y \in A_1$ , which means  $\lambda = 1$  in (2.6), we can write

$$[w(yz)]_{\frac{1}{2}} = [z(wy)]_{\frac{1}{2}}. \text{ We also have } 3[w(yz)]_1 = y(wz)_1 + [z(wy)]_1. \tag{2.7}$$

Since  $w, z \in A_{\frac{\lambda}{2}}$ , switching  $w$  and  $z$  in (2.7), we have

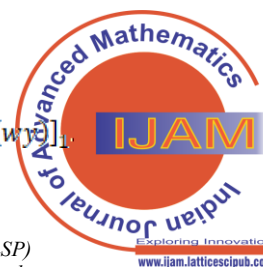
$$3[z(wy)]_1 = y(wz)_1 + [w(yz)]_1. \tag{2.8}$$

The sum of (2.7) and (2.8) gives, after simplifying.

$$[z(wy) + w(yz)]_1 = y(wz)_1. \tag{2.9}$$

Using (2.9) in (2.7) gives  $[w(yz)]_1 = [z(wy)]_1$ . Then, (2.9) can be written.

$$y(wz)_1 = 2[w(yz)]_1 = 2[z(wy)]_1.$$



## Study of Some Degree Five Identities of Type (4,1)

Let's consider  $y = y_\lambda, z = z_\lambda$  and  $w = w_{\frac{1}{2}}$  in each case, that gives result 2).

### Example

Let's consider the five-dimensional commutative  $\mathbb{R}$ -algebra  $A$  in which the non-zero products in the basis  $\{e_1, e_2, e_3, e_4, e_5\}$  are given by  $e_1^2 = e_2, e_1e_2 = e_3, e_2^2 = e_4, e_1e_3 = e_5$ . Then  $A$  satisfies (3), but is not a power associative algebra. In fact, we have  $e_4^2 \neq (e_1^2)^2, A^2 = \langle e_2, e_3, e_4, e_5 \rangle, A^3 = \langle e_3, e_4, e_5 \rangle, A^4 = \langle e_5 \rangle$  and  $A^5 = 0$ . In the complete linearization, each term cancels, then  $A$  satisfies (3).

### Example

Let's consider the five-dimensional commutative  $\mathbb{R}$ -algebra  $A$  in which the non-zero products in the basis  $\{e, a_{\frac{1}{2}}, a_2\}$  is given by

$$e^2 = e, ea_2 = a_{\frac{1}{2}}, ea_{\frac{1}{2}} = \frac{1}{2}a_{\frac{1}{2}}.$$

If 4 or 5 variables are replaced by  $e$ , equation (3) is satisfied. In all other cases, each term cancels. Then,  $A$  satisfies (3).

### Remark

An algebra defined by (3) does not necessarily contain any idempotents. It is well known that the two-dimensional algebra of J. M. Osborn has non-zero products in the basis  $\{s, t\}$  is given by

$$s^2 = s + \frac{1}{2}t, st = \frac{1}{2}t, \text{ does not contain any idempotents.}$$

However, it satisfies (3).

### Note

Following A. A. Albert's study, we can introduce it for all  $x_\lambda \in A_\lambda (\lambda \in \{0,1\})$ , the  $F$  linear map

$$S_{x_\lambda}: A_{\frac{1}{2}} \rightarrow A_{\frac{1}{2}}, \quad y_{\frac{1}{2}} \mapsto x_\lambda y_\lambda \quad (\lambda = 0,1). \quad \text{Since}$$

$$(x_\lambda y_\lambda)x_{\frac{1}{2}} = x_\lambda(y_\lambda x_{\frac{1}{2}}) + y_\lambda(x_\lambda x_{\frac{1}{2}}), \quad \text{which means,}$$

$$S_{x_\lambda} S_{y_\lambda} + S_{y_\lambda} S_{x_\lambda} = S_{x_\lambda} y_\lambda, \quad \text{the following map}$$

$$S_\lambda: A_\lambda \rightarrow \text{End}_F(A_{\frac{1}{2}}), \quad x_\lambda \mapsto 2S_{x_\lambda} \quad \text{satisfies}$$

$S_\lambda(x_\lambda y_\lambda) = S_\lambda(x_\lambda) * S_\lambda(y_\lambda)$ , in which  $f * g = \frac{1}{2}(fg + gf)$  defines the product in  $\text{End}_F(A_{\frac{1}{2}})^+$ . That is,  $S_\lambda$  is a morphism of algebras, whose kernel

$$J_\lambda = \ker S_\lambda = \{x_\lambda \in A_\lambda | x_\lambda A_{\frac{1}{2}} = 0\} \text{ is an ideal of } A_\lambda.$$

Since  $J_\lambda$  is the kernel of a morphism of algebras, of  $A_\lambda$  into the special Jordan algebra  $\text{End}_F(A_{\frac{1}{2}})^+$ , then

$$z = (p^2, q, p) \in J_\lambda, \quad (\lambda = 0, 1), \quad \text{with } (\cdot, \cdot) \text{ being the associator.}$$

## IV. IDENTITY (4)

In this section,  $A$  is an algebra defined by (4). Taking  $x=e$  in (4), we have

$$[L_e(L_e - I)(3L_e^2 - 3L_e + I)](y) = 0,$$

with  $y \in A$ . Let's consider the polynomial  $p(t) = t(t-1)(3t^2 - 3t + 1)$ .

The following lemma is well-known

### Lemma

Let  $A$  be an algebra over a field  $F$  such that every subfield of the centralizer of  $A$  is contained in  $F$  and let's consider  $r$  an element of  $F$  which is not a square in  $F$ . Then, if  $A$  is simple, so is it for  $A_L$ , in which  $L=F(k)$ , with  $k$  such that  $k^2 = r \in F$ .

We shall use the extension of the field  $F$  if necessary, and assume that  $p(t) = t(t-1)(t-\lambda)(t-\bar{\lambda})$ ; Then the Peirce decomposition of  $A$  is

$$A = A_0 \oplus A_1 \oplus A_\lambda \oplus A_{\bar{\lambda}}.$$

Following the study of subsection (1.2), we easily establish this theorem

### Theorem

Let  $A$  be an algebra defined by (4) over a suitable extension of  $F$ . Then the Peirce decomposition of  $A$ , relative to idempotent  $e$  is  $A = A_0 \oplus A_1 \oplus A_\lambda \oplus A_{\bar{\lambda}}$ , with  $A_0$  and  $A_1$  being orthogonal subalgebras,  $A_\lambda A_{\bar{\lambda}} = A_{\bar{\lambda}} A_\lambda = A_\lambda A_{\bar{\lambda}} = 0$  and  $A_i A_\mu \in A_\mu, (i = 0, 1, \mu \in \{\lambda, \bar{\lambda}\})$ . Moreover, we have

- i.  $y_0(z_0 w_\lambda) = z_0(y_0 w_\lambda) = \bar{\lambda} w_\lambda(y_0 z_0)$ ,
- ii.  $y_0(z_0 w_{\bar{\lambda}}) = z_0(y_0 w_{\bar{\lambda}}) = \lambda w_{\bar{\lambda}}(y_0 z_0)$ ,
- iii.  $y_1(z_1 w_\lambda) = z_1(y_1 w_\lambda) = \delta w_\lambda(y_1 z_1)$ ,
- iv.  $y_1(z_1 w_{\bar{\lambda}}) = z_1(y_1 w_{\bar{\lambda}}) = \bar{\delta} w_{\bar{\lambda}}(y_1 z_1)$ , with  $\delta = \frac{1}{26}(19 - i\sqrt{3})$
- v.  $y_0(z_1 w_\lambda) = \varkappa z_1(y_0 w_\lambda)$ ,
- vi.  $y_0(z_1 w_{\bar{\lambda}}) = \bar{\varkappa} z_1(y_0 w_{\bar{\lambda}})$ , with  $\varkappa = -\frac{1}{2}(1 - i\sqrt{3})$ .

## V. CONCLUSION

The study of these degree 5 identities, conducted in the presence of an idempotent, allowed us to determine the relationships between the Peirce components. In each case, we were able to find two orthogonal subalgebras. We also managed to define, in each case, a homomorphism on one of the subspaces of the algebra  $A$ , whose kernel is an ideal of  $A$ . These results allow us to consider further study of these identities when dealing with nil-algebras or baric algebras, as in ([3]) and ([4]).

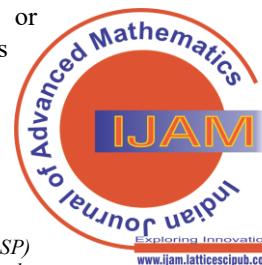
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