Some Results on Specificity of Possibility Distributions

Jayesh V. Karanjgaonkar

***Abstract**: Specificity of a possibility distribution is akin to the entropy of a probability distribution. It serves an essential purpose to zero in on the maximum probability observation. However, when we discuss the existing definition of possibility distribution, it lacks applicability in real-world problems; hence, specificity also becomes an underrated measure for gauging the degree of uncertainty in a possibility distribution. In this paper, we present new findings on the specificity of a possibility distribution, resulting from our research on data-based semantic information analysis in hybrid human-machine systems. In this research, we propose a new frequency-based possibility and probability measure and formalise a new method for fitting restrictions on data or information available in the system. We will demonstrate that the proposed formula is superior to existing specificity measures and discuss various applications of specificity measures in solving problems related to hybrid systems. We shall summarise this paper by providing a real-world application of the proposed measure.*

***Keywords**: Possibility Distribution, Restriction, Specificity, Hybrid System, Semantic Information*

I. INTRODUCTION

Specificity is a key concept in possibility distribution theory, much like entropy is in probability distributions. The idea of specificity is deeply rooted in possibility theory. Specificity of a possibility distribution represents the degree to which a possibility distribution points to a single observation. In possibility theory, a possibility distribution (denoted by π) assigns to each possible world or state a degree of possibility ranging from 0 (impossible) to 1 (fully possible). The specificity of a possibility distribution measures how sharply it distinguishes between possible and impossible states—i.e., how "precise" or "informative" it is. A possibility distribution is specific if it rules out as many states as possible, assigning high possibility (close to 1) to a few states and low possibility (close to 0) to the rest. Conversely, a non-specific (or less specific) distribution spreads the possibility more widely, leaving many states as plausible. For example, a possibility distribution of winning probability {India = 1.0, Australia = 0.5, New Zealand = 0.3, England = 0.2} is more specific, and {India = 0.8, Australia = 0.8, New Zealand = 0.5, England = 0.5}is less specific.

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II. SPECIFICITY MEASURES

Yager [1] proposed the idea of the specificity of a fuzzy set, which is defined by a function $Sp: X \to [0,1]$, having the following properties -

1. $\forall A \subset X, Sp(A) \in [0,1]$.
2. $Sp(A) = 1 \Leftrightarrow A$ is a singleton set in X.
3. $Sp(\phi) = 0$.
4. For $A, B \subset X$, such that $A \subseteq B Sp(A) \geq Sp(B)$.
5. Specificity of a set increases if the maximum membership grade increases, and decreases when any other membership grade increases.

When X is finite, Yager proposed the following formula for the specificity measure

$$Sp(A) = \int_0^{\bar{\alpha}} \frac{1}{Card(A_\alpha)} d\alpha \quad \dots \quad (1)$$

where, $\bar{\alpha} = \max_{x \in X} \mu_A(x)$, $A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$, and $Card(A_\alpha)$ denotes the cardinality of A_α. Yager also defined a new class of specificity measure by the following formula –

$$Sp(A) = w_1\alpha_1 - \frac{1}{n-1}\sum_{j=2}^{n} w_j\alpha_j \quad \dots \quad (2)$$

where α_i is the decreasing set of possibility grades, with α_1 is the most significant possibility.Also w_j are *weights* satisfying the following conditions -

1. $w_j \in [0,1]$
2. $w_1 = 1$
3. $w_i \geq w_j$ if $i \leq j$
4. for all $n \geq 2$, $\sum_{j=2}^{n} w_j \leq 1$

In [2], Dubois and Prade defined the following formulae for the specificity of a normalized fuzzy set A, provided that all n elements of X are arranged in decreasing values of μ_A and $\mu_A(x_{n+1}) = 0$,

$$Sp(A) = \sum_{i=1}^{n} \frac{1}{i}\{\mu_A(x_i) - \mu_A(x_{i+1}) \quad \dots \quad (3)$$

Dubois and Prade also proved a one-to-one correspondence between probability and possibility distribution by following a pair of transforms.

$$\pi_X(x) = \sum_{x' \in X} \min\big(p_X(x'), p_X(x)\big) \quad \dots \quad (4)$$

and its inverse -

$$p_X(x_i) = \sum_{j=i}^{n} \frac{1}{j}\left(\pi_X(x_j) - \pi_X(x_{j+1})\right),$$
$$\text{for } i = 1,2,\dots n \tag{5}$$

The specificity of any set A of X can be seen as the probability of the element which has the most significant membership degree in the above sense of transforms, i.e. $Sp(A)$ is the probability, computed by π_X using the above equations for the most possible values of x. However, the above formulations do not support continuous data analysis, as they require arranging the data in decreasing order of membership grades. Hence, we propose a change in the formulae for the specificity measure by removing the precondition of placing the data set.

III. RELATING INFORMATION, POSSIBILITY AND SPECIFICITY

Possibility conveys two *meanings*- (i)A physical meaning - Ease to achieve an outcome. and (ii) An epistemic meaning - Logically consistent with available information. Zadeh was the advocate of physical sense, which is reflected in the condition of maximality for the possibility measure $Poss(A \cup B) = \max\{Poss(A), Poss(B)\}$ where A and B are disjoint sets [3].

The degree of ease of some decision or action which produces the results A or B is given by the easiest of the two decisions A or B. This idea is reflected in preferences, i.e., when we consider two mutually exclusive alternatives, the one that is most feasible (in any given sense) is usually preferred. The epistemic point of view is practical when incomplete information is present for a decision. Suppose U be a set of discourses and x be some variable, suppose $x \in E$, be the piece of information available, since E is a non-empty, non-singleton set, hence it contains more elements and thus E has more uncertainty about the actual value of x, as it can be any value (but only one value) from E.In this line of thought, we can define a possibility measure by

$$\Pi_E(A) = \begin{cases} 1 & \text{if} A \cup E \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\Pi_E(A) = 1$ implies that $x \in A$ whenever $x \in E$ (because $A \cup E \neq \phi$), similarly $\Pi_E(A) = 0$ implies that $x \notin A$ whenever $x \in E$.

The following are two scenarios attached to a possibility distribution -

A. *Complete Knowledge* - Suppose $E = u_0$, for some u_0, then $\Pi_E(A) = N_E(A) = 1$, iff$u_0 \in A$ i.e., there is one and only one event that is possible and certain.

B. *Complete Ignorance* - If $E = U$, then for each $A \neq \phi$, $\Pi_E(A) = 1$ and for each $A \neq U$, $N_E(A) = 0$, i.e. everything is possible and nothing is specific.

Any kind of data collection for a semantic information-based decision problem has two essential characteristics [4] -

1. Specificity, which provides an interval estimate for the data set. An extensive set is less specific (more uncertain) but better for approximation.
2. Entailment principle, we can always predict a larger set than the actual fuzzy set. If $x \in A$ and $A \subset F \subset G$, then we can predict the set G with more accuracy than set F.

For example, "today is a hot day" can be translated as an interval of values of a variable temperature. The set can be large (less specific), hence there will be a higher chance of representing a hot day. However, as the number of observations increases, more and more frequencies will become equal, and the set will become uniformly distributed, making it impossible to select any particular value.

When we reduce the number of observations, the set becomes more specific. Still, there is a chance that the values do not accurately represent the correct temperature to qualify as hot. Thus, a large set is better for prediction; i.e., lowering specificity increases the chance of choosing the proper values of the variable. However, this characteristic works against Maximum specificity, because enlarging the prediction set will reduce the specificity of the estimation. In approximate reasoning with possibilistic distributions, we must work against both of the above theories. We have to predict the correct value with maximum specificity. In statistics, the theory of sampling errors operates on the same basic principle, based on probability distributions.

IV. A NEW FORMULATION FOR POSSIBILITY

Our work in semantic information analysis for a hybrid man-machine system follows a modified version of Zadeh's Information Principle [5]. The basic methodology followed in this process is that the data observed by the machine sub-part is converted into a normalised possibility distribution, which is then used in decision-making, prediction, and description of the system. For the analysis of any stream of data, we focus on the mode frequency because it is the most probable and most likely frequency in any sample [6]. Suppose $X = \{x_1, x_2, x_3 \dots x_n\}$ be the set of observations or values of the decision variable. Let $F = \{f_1, f_2, f_3 \dots f_n\}$ be the respective frequencies. Together X and F constitutes our version of *explanatory database* or ED [6]. Suppose that the ED is uni-modular and the mode observation, frequency pair is denoted by (x_M, f_M). It is also assumed that the mode frequency is much higher than other frequencies. The probability measure, termed *natural probability*, $P: (X, F) \to [0,1]$ by –

$$P(x_i) = \frac{f_i}{N} \tag{6}$$

Next, we define the possibility measure, termed *natural possibility*, $\pi_M: (X, F) \to [0,1]$ by –

$$\pi_M(x_i) = \frac{f_i}{f_M} \tag{7}$$

Natural possibility distribution defined by 4.2 converts the ED. (X, F) into a possibility distribution, given the condition of a unique higher mode. We have also defined a probability measure to complement the possibility measure, as well as a proximity value measure. The following table denotes various measures defined to approximate the distribution into an optimal possibility distribution [6].

Table-I: Various Measures for Possibilistic Analysis, where $N = \sum_{i=1}^{n} f_i$ and $\bar{N} = N - f_M$.

Measure	Formula
Natural Probability (P)	$P(x_i) = \frac{f_i}{N}$
Natural Possibility(π_M)	$\pi_M(x_i) = \frac{f_i}{f_M}$
Proximity Value (PV)	$\sum_{i=1}^{n} \pi_M(x_i) - \sum_{i=1}^{n} P(x_i) = \frac{\bar{N}}{f_M}$
Elemental Proximity Value (EPV)	$\pi_M(x_i) - P(x_i) = (\frac{\bar{N}}{N f_M}) f_i$
Mode Proximity (MPV)	$\pi_M(x_M) - P(x_M) = \frac{\bar{N}}{N}$
Probability-Possibility Consistency Principle	$\gamma = \frac{1}{N f_M} \sum_{i=1}^{n} f_i^2$
Probability to Possibility conversion	$\pi_M(x_i) = \frac{P(x_i)}{P(x_M)}$
Possibility to Probability conversion	$P(x_i) = \frac{\pi_M(x_i)}{\sum_{i=1}^{n} \pi_M(x_i)}$

V. PROPOSING NEW FORMULAE FOR SPECIFICITY

We shall use natural possibility and probability as the input in the following formula defined by Yager [1]

$$Sp(A) = w_1 \alpha_1 - \frac{1}{n-1} \sum_{j=2}^{n} w_j \alpha_j \tag{8}$$

where $w_i \in (0,1)$ are weights with $w_1 = 1$ and decreasing values as i increases.The α_1 is the most significant possibility. We replace the respective weights w_i by probability values $P(x_i)$ and we define $\alpha_j = \pi_M(x_j)$, with $\alpha_1 = \pi_M(x_M) = 1$ (most significant possibility). Using our defined measure of natural possibility, the specificity becomes -

$$Sp(X) = P(x_M)\pi_M(x_M) - \frac{1}{n-1} \sum_{x_j \neq x_M} P(x_j)\pi_M(x_j)$$

$$= \frac{f_M}{N} \frac{f_M}{f_M} - \frac{1}{n-1} \sum_{f_j \neq f_M} \frac{f_j}{N} \frac{f_j}{f_M}$$

$$Sp(X) = \frac{f_M}{N} - \frac{1}{n-1} \sum_{f_j \neq f_M} \frac{f_j^2}{N f_M} \tag{9}$$

Above formula defines the specificity of the explanatory database (X, F)through natural possibilities and probability distributions. We have replaced the weights w_i by the probability of each element $P(x_i)$, save w_1, which is not equal to 1 but takes the value of the mode probability, i.e. $P(x_m)$, so as $\sum_{j=2}^{n} w_j \leq 1$The departure from the formula of specificity defined by Yager is that it can be applied to any frequency-based setting. Here, the maximum specificity will be 1 only when $f_M = N$, and every other $f_j = 0$. As the uniformity increases in the sample (non-modal frequencies increase, keeping the sum of frequencies N and the number of observations n fixed), the specificity decreases. It reaches zero when –

$$Sp(X) = 0 = \frac{f_M}{N} - \frac{1}{n-1} \sum_{f_j \neq f_M} \left(\frac{f_j^2}{N f_M} \right)$$

$$\frac{f_M}{N} = \frac{1}{n-1} \sum_{f_j \neq f_M} \frac{f_j^2}{N f_M}$$

$$= \frac{1}{n-1} \left[\frac{f_1^2}{N f_m} + \frac{f_2^2}{N f_m} + \ldots \frac{f_n^2}{N f_m} \right]$$

$$\frac{f_M}{N} = \frac{1}{(n-1)N} \left[\frac{f_1^2}{f_m} + \frac{f_2^2}{f_m} + \ldots \frac{f_n^2}{f_m} \right]$$

$$1 = \frac{1}{n-1} \left[\left[\frac{f_1}{f_m} \right]^2 + \left[\frac{f_2}{f_m} \right]^2 + \ldots \left[\frac{f_n}{f_m} \right]^2 \right]$$

$$\frac{f_M}{f_M} = \frac{1}{n-1} [[\pi_M(x_1)]^2 + [\pi_M(x_2)]^2 + \ldots [\pi_M(x_n)]^2]$$

$$\left[\frac{f_M}{f_M} \right]^2 = [\pi_M(x_M)]^2$$

$$= \frac{1}{n-1} [[\pi_M(x_1)]^2 + [\pi_M(x_2)]^2 + \ldots [\pi_M(x_n)]^2]$$

$$[\pi_M(x_M)]^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} [\pi_M(x_j)]^2 \tag{10}$$

The above equation provides the limiting values of the sum of squared possibilities, so that the specificity of a member of ED becomes zero. Since $[\pi_M(x_M)] = 1$ hence we get

$$\sum_{x_j \neq x_M} [\pi_M(x_j)]^2 = n - 1 \tag{11}$$

This provides another condition for zero specificity. Here, the frequency distribution (X, F) is not a uniform distribution, but a non-specific one, i.e., the mode frequency constitutes one portion of the distribution, and the remaining frequencies constitute the reduced portion of the possibility distribution. Recall that the sum of the squared possibilities of a uniform distribution will be n,, the i.e. number of observations, the above equation provides a condition of zero specificity, by limiting the squared reduced sum to $n - 1$ Also, since we can put $P(x_i) = \frac{\pi_M(x_i)}{\sum_{i=1}^{n} \pi_M(x_i)}$, we get

$$[\pi_M(x_M)]^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} [\pi_M(x_j)]^2$$

$$[\pi_M(x_M)]^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} \left[\frac{f_j}{f_M} \right]^2 \tag{5.2}$$

$$[\pi_M(x_M)]^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} \left[\frac{f_j}{f_M} \times \frac{N}{N} \right]^2$$

$$[\pi_M(x_M)]^2 = \frac{1}{n-1} \left(\frac{N}{f_M} \right)^2 \sum_{x_j \neq x_M} \left[\frac{f_j}{N} \right]^2$$

$$[\pi_M(x_M)]^2 \times \left(\frac{f_m}{N} \right)^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} \left[\frac{f_j}{N} \right]^2$$

Putting $\pi_M(x_M) = 1, \frac{f_j}{N} = P(x_j)$ and$\frac{f_m}{N} = P(x_M)$, we get

$$(P(x_M))^2 = \frac{1}{n-1} \sum_{x_j \neq x_M} [P(x_j)]^2$$

which provides an analogous formula for values of probability at zero specificity. In case of a uniform distribution, i.e. $\pi_M(x_i) = 1$

and $P_M(x_i) = \frac{1}{n}$, using formulae 5, we get -

$$Sp(X) = P(x_M)\pi_M(x_M) - \frac{1}{n-1}\sum_{x_j \neq x_M} P(x_j)\pi_M(x_j)$$

$$= \frac{1}{n} \times 1 - \frac{1}{n-1}\sum_{f_j \neq f_M} \frac{1}{n}$$

$$= \frac{1}{n} - \frac{1}{n-1} \times \frac{n-1}{n}$$

$$= 0$$

Thus, for a uniform distribution, the above-defined specificity measure 5 becomes zero, i.e., there is no specificity for a uniform distribution. Hence, in our formulation, the specificity of the possibility distribution can be zero when $\sum_{x_j \neq x_M} [\pi_M(x_j)]^2 = n - 1$ or the distribution is uniform. Since the natural possibility distribution is defined based on the unique mode, we shall use the criterion of reduced squared sum in our decision process.

VI. EXAMPLE OF TEMPERATURE OF A DAY

We have taken a natural language statement *Today is a hot day*. The resulting possibilistic restriction is (as based on context) temperature (X) is Hot (R), where X is a set of real numbers and R is a fuzzy set. The membership grades denote the possibility distribution for the day's temperature. We shall convert this problem into a decision problem for a hybrid system. Suppose a machine must decide on an ambient temperature based on natural language statements from human communication. Consider the data shown in the following table-

Table-II: Temperature of Month June 2024

Max. Temp.	Days	Min. Temp.	Days
27	1	23	10
28	5	24	6
29	3	25	2
30	3	26	6
31	2	27	5
32	2	28	1
33	3		
34	1		
35	3		
36	1		
37	3		
38	2		
39	1		

The above frequency distribution satisfies the requirement of a unique high mode. Hence, using the *Natural Possibility Measure*, defined by $\pi_M(x_i) = \frac{f_i}{f_M}$ and *Natural Probability Measure*, defined by $P(x_i) = \frac{f_i}{N}$, we get the following tables-

Table-III: Characteristic Calculation

Max. Temp.					
T	**F**	**π_M**	**P**	**EPV**	**EPV**
27	1	0.2	0.03	0.17	0.17
28	5	1	0.17	0.83	0.83
29	3	0.6	0.1	0.5	0.5
30	3	0.6	0.1	0.5	0.5
31	2	0.4	0.07	0.33	0.33
32	2	0.4	0.07	0.33	0.33
33	3	0.6	0.1	0.5	0.5
34	1	0.2	0.03	0.17	0.17
35	3	0.6	0.1	0.5	0.5
36	1	0.2	0.03	0.17	0.17
37	3	0.6	0.1	0.5	0.5
38	2	0.4	0.07	0.33	0.33
39	1	0.2	0.03	0.17	0.17

Min. Temp.				
T	**F**	**π_M**	**P**	**EPV**
23	10	1	0.33	0.67
24	6	0.6	0.20	0.40
25	2	0.2	0.07	0.13
26	6	0.6	0.20	0.40
27	5	0.5	0.17	0.33
28	1	0.1	0.03	0.07

From the above table, the characteristic values of the above distribution are -

Table-IV: Characteristics of Distribution

Characteristic	Max. Temp. Value	Min. Temp. Value
MPV	0.83	0.67
PV	5.0	2.0
Average	$T_R = 34°C$ with EPV = 0.17	$T_R = 25°C$ with EPV = 0.13
Specificity	0.1328	0.2653

VII. RESULT

In the above example, we have shown a fundamental calculation of instantiation and precisiation of a possibility distribution. It is clear that -

- The maximum temperature group has more observation, however they occur in similar frequencies. On the other hand, the minimum temperature group has fewer observations, but they appear at different frequencies.
- Similarly, the minimum temperature group is more specific than the maximum temperature group, because the number of distinct observations is less in the minimum temperature group. temperature group, and the mode frequency is higher in mini. temperature group.

VIII. CONCLUSION

As is clear from the above example, constructing a possibility distribution using context variables and calculating various characteristics is straightforward and can be applied to multiple data settings. Similarly, the calculation of specificity does not require a specific arrangement of frequency or probability distribution; therefore, it can be applied to many more situations involving continuous data streams or fixed datasets. Our formulation of specificity is less restrictive but efficient in the application of various possible scenarios.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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