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On Algebraic and Topological Properties of Some Fundamental Groups

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Abstract: ***In this paper, we study the algebraic and topological properties of some topological spaces. We note that the fundamental group of a topological group is abelian and we study some spaces of the same homotopy type with the unit circle S^1. The basic group of the unit circle S^1 is isomorphic to the additive group of integers. We say that a topological space is simply connected if it is path-connected and has a trivial fundamental group. We show that the fundamental group of a n-punctured plane is free and we characterize some surfaces as topologically distinct.***

Keywords: Homeomorphism, Path, Loop, Topological Group, Fundamental Group, Free Product.

I. INTRODUCTION

It's well known that a simple connexity is a homotopy invariant. If X is a simply connected space, then any two paths having the same initial and final points are path homotopic [1]. If X and Y are two connected spaces by arcs and have the same homotopy type, their fundamental groups are isomorphic [2]. A fundamental group is invariant under homeomorphisms and is a topological invariant [9]. The utility of the group concept in homotopy theory is increased by the relations between the fundamental group considered as a functor from based topological spaces to groups

$\pi_1 : Top_* \longrightarrow Groups$

and another functor called the classifying space

$B: Groups \longrightarrow Top_*$

which is the composite of the geometric realization and the nerve functor N from Groups to simplicial sets [3].

Some Problems:

- **Problem** 1: If X is a topological space, under what condition its fundamental group becomes abelian?
- **Problem** 2: Let $p_1,p_2,...,p_n$ be n distinct points in $\mathbb{R}^2$. What is the fundamental group of the n-punctured plane $\mathbb{R}^2 - \{p_1,...,p_n\}$?
- **Problem** 3: What are the spaces that are of the same homotopy as S^1?
- **Problem** 4: Are there surfaces or varieties that are topologically distinct?

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We will provide solutions to these various problems by drawing inspiration from the main results established in the preliminaries [10]. The Seifert-Van Kampen's Theorem gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known [4]. Let R be a subset of a space X; If R is a deformation retract of a space X, then their fundamental groups are isomorphic [5]. It's known that the circle.

$C = \{(x,0) : 0 \leq x \leq a\}$

is a deformation retract of the Mobius band

$M = \{(x,y) : 0 \leq x \leq a, -1 \leq y \leq 1\}$ [5].

As set C is homeomorphic to the unit circle and the unit circle S^1 has the set of integers Z as the fundamental group we see that the fundamental group of the Mobius band is $\mathbb{Z}$ by [5], [6].

II. PRELIMINARIES

We recall here some useful definitions and results.

A. Fundamental group

Definition 2.1. *Let $I = [0,1]$ be the unit interval, and let X be a topological space. If $x_0 \in X$, a pointed space (X,x_0) is a space X together with x_0, and x_0 is called the basepoint of X. A path is a continuous map $f: I \to X$. A loop in a pointed space (X,x_0) is a path $f: I \to X$ such that $f(0) = f(1) = x_0$.*

Definition 2.2. *Let X and Y be topological spaces. Let $f: X \to Y$ and $g: X \to Y$ be two continuous functions. We say that f and g are homotopic if there exists a continuous function $F: X \times I \to Y$ such that $\forall x \in X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We denote that f and g are homotopic and F is called a homotopy between f and g, and we write $f \sim g$.*

Proposition 2.1. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

Definition 2.3. *A homotopy equivalence is a continuous map of topological spaces $f: S \longrightarrow T$ such that there exists a continuous map*

$g: T \longrightarrow S$ *with*

$g \circ f \sim Id_S$, $f \circ g \sim Id_T$.

g is called the homotopy equivalence of f.

Proposition 2.2. *Any homeomorphism is a homotopy equivalence.*

Proof. If $f: S \longrightarrow T$ is a *homeomorphism*, then f has a continuous inverse $g: T \longrightarrow S$, so $g \circ f = I_S$ and $f \circ g = I_T$, hence $g \circ f \sim Id_S$ and $f \circ g \sim Id_T$.

Definition 2.4. (*The composition of paths*) *Let f,g: $I \rightarrow X$ be two paths. Define the composition of f and g denoted f.g: $I \rightarrow X$, by*

(f.g)(t)=f(2t) if $0 \leq t \leq \frac{1}{2}$

(*f.g*)(*t*) $=g(2t-1)$ if $\frac{1}{2} \leq t \leq 1$

(2.1)

Remark 2.1. *We denote the equivalence class of a path f under the equivalence relation of homotopy by* [*f*], *and the product operation f.g in* (2.1) *respects homotopy classes since* f_0 ' f_1 *and* g_0 ' g_1 *via homotopies* f_t *and* g_t *respectively, and if* $f_0(1) = g_0(0)$ *so that* $f_0.g_0$ *is defined, then* $f_t.\ g_t$ *is defined and provides homotopy* $f_0.g_0 \sim f_1.g_1$.

Definition 2.5. *Let f: $I \rightarrow X$ be a path with the same starting and ending point* $f(0) = f(1) = x_0 \in X$. *Such a path is called a loop. The common starting and ending point* x_0 *for a class of paths is referred to as the base point. The set of all homotopy classes* [*f*] *of loops f: $I \rightarrow X$ at the basepoint* x_0 *is denoted* $\pi_1(X,x_0)$.

Proposition 2.3. *Let* $\pi_1(X,x_0)$ *be the set of all homotopy classes* [*f*] *of loops f: $I \rightarrow X$ at the base point* x_0. *Then* $\pi_1(X,x_0)$ *is a group concerning the product* [*f*][*g*] = [*f.g*]. *This group is called the fundamental group of X at the point base* x_0.

Definition 2.6.

(*Topological group*) *Let* (*G*,.) *be a group whose law is denoted multiplicatively. We say that G is a topological group if G is a topological space and is endowed with a topology, such that multiplication* $G \times G \longrightarrow G$, $(x,y) \mapsto xy$ *and inversion* $G \longrightarrow G$, $x \mapsto x^{-1}$ *are continuous.*

Definition 2.7. *Let* x_0 +++++ *and* x_1 *be two base points that lie in the same path component of X. Let h: $I \longrightarrow X$ be a path from* x_0 *to* x_1, *with an inverse path*

$\bar{h}(s) = h(1-s)$ *from* x_1 *to* x_0. *For each loop f based at* x_1, *we can associate the*

Loop $h f \bar{h}$. *We can define a change-of-basepoint map*

$\beta_h : \pi_1(X,x_1) \rightarrow \pi_1(X,x_0)$ *by*

$\beta_h(\mathrm{f})=[\mathrm{hf}\bar{h}]$

Proposition 2.4. *Let* x_0 *and* x_1 *be two base points of X. Then, the map* β_h: *is an isomorphism.*

Proof. It is clear that β_h is a homomorphism since $\beta_h[fg]=[\mathrm{hfg}\bar{h}]=[\mathrm{hf}\bar{h}\mathrm{hg}\bar{h}]=\beta_h(f)\beta_h(g)$. We know that β_h is a bijection with inverse β_h since $\beta_h\beta_{\bar{h}}[f] = \beta_h[\bar{h}fh] = [h\bar{h}fh\bar{h}] = [f]$

and similarly $\beta_{\bar{h}}\beta_h[f] = [f]$.

Proposition 2.5. *Let X, and Y be two topological spaces. Suppose*

f: $X \longrightarrow Y$ is a homeomorphism defined by $x \mapsto y$. Then, $\pi_1(X,x)$ *and* $\pi_1(Y,y)$ *are isomorphic.*

Proof. We define a homomorphism $\pi_1(f)$: $\pi_1(X,x) \longrightarrow \pi_1(Y,y)$ by sending a path γ: $I \longrightarrow X$ to the path $f \circ \gamma$: $I \longrightarrow Y$.

Since *f* is a homomorphism, *f* has an inverse map f^{-1}, and the map $\pi_1(f^{-1})$: $\pi_1(Y,y) \longrightarrow \pi_1(X,x)$ is an inverse to $\pi_1(f)$. □

Definition 2.8. *Let X, and Y be topological spaces. X and Y are said to be homotopic, written as $X \sim Y$, if there exist mappings f: $X \longrightarrow Y$ and g: $Y \longrightarrow X$ such that $f \circ g$: $Y \longrightarrow Y$ and $g \circ f$: $X \longrightarrow X$ are homotopic to the identity. We say that X and Y are of the same homotopy type. The map f is called the homotopy equivalence and g, is its homotopy inverse.*

Remark 2.2. *If X and Y are homeomorphic, then X and Y are of the same homotopy type but the converse is not necessarily true. For example, a point {P} and the real line $\mathbb{R}$ are of the same homotopy, type but {P} and* R *are not homeomorphic.*

Proposition 2.6. *"Of the same homotopy type" is an equivalence relation in the set of topological spaces.*

Proof. [7]

Proposition 2.7. *Let*

X, Y be two spaces. Let $x_0 \in X$ *and* $y_0 \in Y$.

Then

$$\pi_1(X \times Y(x_0,y_0)) \cong \pi_1(X,x_0) \times \pi_1(Y,y_0).$$

Proof. Given a loop *f* in *X* and a loop *g* in *Y*, we obtain a loop in $X \times Y$ by viewing *f* as a loop in $X \times \{y_0\}$ and *g* as a loop in $\{x_0\} \times Y$ and taking *f.g* as a loop in $X \times Y$. To show this map is surjective if we have a loop *h* in $X \times Y$, one can write this as a function $s \mapsto h_s(x,y) =$

$(f_s(x), g_s(y))$. Then h_s is homotopic to the composition of the loops $s \mapsto (f_s(x), y_0)$ and $s \mapsto (x_0, g_s(y))$ (where the composition is taken in the fundamental group via the operation of Proposition 2.3).

Example 2.1. *Let* S^1 *be the unit circle. Then* $\pi_1(S^1 \times S^1,(x_0,y_0)) = \pi_1(S^1,x_0) \times \pi_1(S^1,y_0)$.

B. Deformation Retract

Definition 2.9. *Let R be a nonempty subspace of X. If there exists a continuous map f: $X \longrightarrow R$ such that $f|_R = Id_R$, R is called a retract of X and f a retraction.*

Definition 2.10. *Let R be a subspace of X. If there exists a continuous map H: $X \times I \longrightarrow X$ such that*

$H(x,0) = x$ $H(x,1) \in R$ *for any* $x \in R$ (2.2)

$H(x,t) = x$ *for any* $x \in R$ *and any* $t \in I$ (2.3)

The space R is said to be a deformation retract of X. H is therefore a homotopy between Id_X *and the retraction f: $X \longrightarrow R$.*

Proposition 2.8. *Let R be a subspace of X. Since X and R are of the same homotopy type, we have* $\pi_1(X,\ a) \cong \pi_1(R,\ a)\ \forall a \in R$ (2.4)

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Example 2.2. *Let X be the unit circle and Y be the annulus,*

$X = \{e^{i\theta} \mid 0 \le \theta < 2\pi\}$ (2.5)

$Y = \{re^{i\theta} \mid 0 \le \theta < 2\pi, \frac{1}{2} \le r \le \frac{2}{3}\}$ (2.6)

The circle X is a deformation retract of the annulus Y.
Define $f: X \to Y$ *by* $f(e^{i\theta}) = e^{i\theta}$ *and* $g: Y \to X$ *by* $g(re^{i\theta}) = e^{i\theta}$. *Then* $f \circ g: re^{i\theta} \to e^{i\theta}$ *and* $g \circ f: e^{i\theta} \to e^{i\theta}$. *Observe that* $f \circ g \simeq Id_Y$ *and* $g \circ f \simeq Id_X$

There exists a homotopy

$H(re^{i\theta}, t) = \{1 + (r-1)(1-t)\}\, e^{i\theta}$

Which interpolates Id_X *and* $f \circ g$, *keeping the points on X fixed. Hence, X is a deformation retract of Y. As for the fundamental groups we have*

$\pi_1(X,a) \cong \pi_1(Y,a)$

where $a \in X$

Proposition 2.9. *If a space X retracts onto a subspace A, then the homeomorphism* $i_*: \pi_1(A,x_0) \longrightarrow \pi_1(X,x_0)$ *induced by the inclusion* $i: A \hookrightarrow X$ *is injective. If A is a deformation retract of X, then* i_* *is an isomorphism.*

C. Contactable Spaces

Definition 2.11. *Let X be topological space and let* $a \in X$. *If a is a deformation retract of X, X is said to be contractible.*

Remark 2.3. *Let* $c_a: X \longrightarrow \{a\}$ *be a constant map. If X is contractible, there exists a homotopy* $H: X \times I \longrightarrow X$ *such that.*

$H(x,0) = c_a(x) = a$ *and* $H(x,1) = Id_X(x) = x$ *for any* $x \in X$, *and* $H(a,t) = a$ *for any* $t \in I$. *The homotopy H is called the contraction.*

Example 2.3. *The space* $X = \mathbb{R}^n$ *is contractible to the origin 0. We can define* $H: \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$ *by* $H(x,t) = tx$. *Then we obtain* $H(x,0) = 0$ *and* $H(x,1) = x$ *for any* $x \in X$ *and* $H(0,t) = 0\ \forall\, t \in I$.

Theorem 2.1 *Let X be a contractible space. Its fundamental group is trivial,* $\pi_1(X,x_0) \cong \{e\}$. *In particular, the fundamental group of* $\mathbb{R}^n$ *is trivial,* $\pi_1(\mathbb{R}^n,x_0) \cong \{e\}$ *for all* $n \ge 1$.

Proof. Since a contractible space has the same fundamental as a *point p* and a point has a trivial fundamental group, $\pi_1(X,x_0) \cong \{e\}$ by Proposition 2.9.

D. The Fundamental Group of the Circle S^1

We have the following theorem which provides the fundamental group of the the 1-sphere *considered* as a unit circle S^1 on the complex plane.

Theorem 2.2. *The fundamental group of* S^1 *is isomorphic to* $\mathbb{Z}$.

$\pi 1(S1) \sim= Z$. (2.7)

Proof. Cf. [8]

E. Product of Spaces

Theorem 2.3. *Let X and Y are arcs connected to topological spaces. Then*

$\pi_1(X \times Y,(x_0,y_0))$ *is isomorphic to* $\pi_1(X,x_0) \oplus \pi_1(Y,y_0)$.

Proof. Define projections $p_1: X \times Y \longrightarrow X$ and $p_2: X \times Y \longrightarrow Y$. If α is a loop in $X \times Y$ at (x_0,y_0), then $\alpha_1 \equiv p_1(\alpha)$ is a loop in X at x_0, and $\alpha_2 \equiv p_2(\alpha)$ is a loop in Y at y_0. Conversely, any pair of loops α_1 of X at x_0 and α_2 of Y at y_0 determines a unique loop $\alpha = (\alpha_1,\alpha_2)$ of $X \times Y$ at (x_0,y_0). Define a homomorphism $\varphi: \pi_1(X \times Y,(x_0,y_0)) \longrightarrow \pi_1(X,x_0) \oplus \pi_1(Y,y_0)$ by $\varphi([\alpha]) = ([\alpha_1],[\alpha_2])$.

By construction φ has an inverse, hence it is the required isomorphism, and $\pi_1(X \times Y,(x_0,y_0)) \cong \pi_1(X,x_0) \oplus \pi_1(Y,y_0)$.

F. Some Examples

1. Let $T^2 = S^1 \times S^1$ be a torus. Then $\pi_1(T^2) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

Similarly, for n-dimensional torus $T^n = S^1 \times S^1 \times \ldots \times S^1$ n-times we have

$\pi_1(T^n) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ n-times

2. Let $X = S^1 \times \mathbb{R}$ be an infinite cylinder. Since $\pi_1(\mathbb{R}) \cong \{e\}$, we obtain $\pi_1(X) \cong \mathbb{Z} \oplus \{e\} \cong \mathbb{Z}$.
   because the cylinder has S^1 as a deformation retracted by the homotopy

$H((x,y),t) = (x,(1-t)y)$

The same is true for a compact cylinder

$C = S^1 \times I$ for $I = [0,1]$, by the same previous homotopy, so $\pi_1(C) = \mathbb{Z}$.

3. For the space $S_1 = \{x \in \mathbb{R}^2 : |x| > 1\}$ we have $\pi_1(S_1) = \mathbb{Z}$. The space $2S_1 = \{x \in \mathrm{R}^2 : |x| = 2\}$ is a deformation retract of S_1 by the straight-line homotopy

   $H(x,t) = (1-t)x + \dfrac{2tx}{|x|}$.

4. The space $S_2 = \{x \in \mathbb{R}^2 : |x| \ge 1\}$ has the unit circle S^1 as deformation retract by the straight-line homotopy

   $H(x,t) = tx$

   $(1-t)x + \frac{tx}{|x|}$

   So $\pi_1(S_2) \cong \mathbb{Z}$.

5. The fundamental group of solid torus $S^1 \times B^2$ is

   $\pi_1(S^1 \times B^2) = \pi_1(S^1) \times \pi_1(B^2) = \mathbb{Z} \times \{e\} \cong \mathbb{Z}$

6. The product space $S^1 \times S^2$ has a fundamental group

   $\pi_1(S^1 \times S^2) = \pi_1(S^1) \times \pi_1(S^2) = \mathbb{Z} \times \{e\} \cong \mathbb{Z}$.

7. Let us prove that the fundamental group of the real projective plane $\mathbb{R}P^2$ is isomorphic to $\mathbb{Z}_2$ a cyclic group of order 2, and likewise for real projective n-space $\mathbb{R}P^n$.

The projective plane P^2 is a surface, and the quotient map $p: S^2 \longrightarrow P^2$ defined as $p(x) = [x] = \{-x,x\}$ is a covering map.

Note that n-space P^n can be similarly defined by "identifying" x and $-x$ for each x on the n-sphere S^n. The 2 *sphere* is simply connected since the map

$$R: \mathbb{R}^3 - \{0\} \longrightarrow S^2$$

given by $x \mapsto \frac{x}{IxI}$

is a deformation retract of $\mathbb{R}^3 - \{0\}$ onto the 2-sphere. The projective plane $\mathbb{R}P^2$ has the fundamental group $\mathbb{Z}/2\mathbb{Z}$ since it is the quotient of S^2 by the identification $x = -x$. The projection map is a covering map, and the group of covering transformations is exactly $\mathbb{Z}_2 = \{Id, x{=}-x\}$.

The nontrivial element in the fundamental group of $\mathbb{R}P^2$ can be thought of as the quotient of a great chord on S^2 that connects the North to the South Pole [11].

G. Covering Spaces

As we will see, covering spaces can be used to compute fundamental groups of some spaces.

Definition 2.12. *Let A and B be two spaces. A map* p: $A \longrightarrow B$ *is a covering space if: around each point* $b \in B$, *there is a neighborhood N of b so that* $p^{-1}(N)$ *is a disjoint union of sets* A_i *each of which is mapped homomorphically onto N by p.*

Example 2.4. *The classic example is the exponential map exp: exp*: $\mathbb{R} \longrightarrow S^1$.

Theorem 2.4.

(*Classification of Covers*)

Let B be a space. To every subgroup of $\pi_1(B,b)$ *there is a covering space of B so that the induced subgroup is the given one.*

Remark 2.4.

- *If* (A, a,p) *and* (A', a',p') *are two different covers corresponding to the same subgroup, then there exists a unique homeomorphism* $h : (A, a) \longrightarrow (A', a')$ *so that* $p = ph$.
- *Note that a cover of a cover is a cover so that smaller subgroups correspond to "higher" covers.*
- *The trivial subgroup corresponds to the "universal cover" of B.*
- *Note too, that if* $p : (A, a) \longrightarrow (B,b)$ *is a universal cover, then for any* $a' \in p^{-1}(b)$, *there is a unique homeomorphism* $h_{a'}: A \longrightarrow A$

  so that

  $h_{a'}(a)$=a'

  $ph_{a'} = p$.

The set of homeomorphisms that satisfy the second condition form a group (sometimes called the group of covering transformations or the group of deck transformations). It turns out that this group is isomorphic to the fundamental group of B.

H. The Wedge Sum

A topological space can be presented in the form of a wedge with two or *several* circles. We give here the definition of the wedge of circles.

Definition 2.13. *Let* (X,x_0) *and* (Y,y_0) *are two-based spaces. Their wedge sum, or one-point union, is*

$$X \vee Y = X \sqcup Y / \sim$$

where $x_0 \sim y_0$ *and* $X \sqcup Y$ *design the disjoint union of X and Y.*

Definition 2.14. *Let X be a space that is a union of the subspaces* S_α, *for* $\alpha \in J$, *each of which is homeomorphic to the unit circle. Assume there is a point p of X such that* $S_\alpha \cap S_\beta = \{p\}$ *whenever* $\alpha \neq \beta$. *If the topology of X is coherent with the subspaces* S_α, *then X is called the wedge of the circles* S_α.

I. Free Product of Groups

Remark 2.5. (*Free group*)

- *Let* G be a group noted as a product, 1 is the *identity element and*

the inverse of each element x is noted as x^{-1}. *If* $\{G_\alpha\}_\alpha \in J$ *is a family of subgroups of G. We say that these groups generate G if every element x of G can be written as a finite product* $x = x_1x_2...x_n$, *where* $x_\alpha \in G_\alpha$. *The sequence* $(x_1,...,x_n)$ *is called a word of length n and it is said to represent the element x of G* [12].

- *However, if* x_i *and* x_{i+1} *both belong to the same subgroup* G_α, *we group them, thereby obtaining the word*

$(x1,...,xi-1,xixi+1,xi+2,...,xn)$

Of length n − 1, which also represents x. Furthermore, if any x_i *equals 1, we can delete* x_i *from the sequence, again obtaining a shorter word that represents x. By successively repeating these reduction operations, one can obtain a word representing x of the form* $(y_1,...,y_m)$ *where no group* G_α *contains both* y_i *and* y_{i+1} *and where* $y_i 6=$ *1 for all i. Such a word is called a reduced word.*

We obtain the following definition.

Definition 2.15. *Let G be a group, and let* $\{G_\alpha\}_{\alpha\in J}$ *be a family of subgroups of G that generate G. Suppose that* $G_\alpha \cap G_\beta$ *consists of the identity element alone whenever* $\alpha \neq \beta$. *We say that G is the free product of the group* G_α *if, for each* $x \in G$, *there is only one reduced word in the group* G_α *that represents x. In this case, we write*

$$\mathbf{G} = \prod\nolimits^*_{\alpha\in J} \boldsymbol{G_\alpha}$$

or in finite case, $G = G_1 * G_2 * ... * G_n$.

Remark 2.6. *An important property of the free product* $*_\alpha G_\alpha$ *is that any collection homomorphisms* φ_α: $G_\alpha \longrightarrow H$ *of groups extends uniquely to a homomorphism* φ: $*_\alpha G_\alpha \longrightarrow H$. *For example, for a free product* $G * H$ *the inclusions* $G \longrightarrow G \times H$ *and* $H \longrightarrow G \times H$ *induce a surjective homomorphism* $G * H \longrightarrow G \times H$.

J. Seifert-Van Kampen Theorem

We quote here the Seifert-Van Kampen Theorem which will allow us to determine the fundamental groups of some spaces.

Theorem 2.5. (*SeiferVan Kampen's Theorem*)

Let $X = \cup_{\alpha=1}^{n} A_\alpha$ *be the union of n path connected, such that each* $A_\alpha \cap A_\beta$ *is path connected, and where each* A_α *contains a given basepoint* $x_0 \in X$. *We have homomorphisms* $\pi_1(A_\alpha,x_0) \longrightarrow \pi_1(X,x_0)$ *induced by the inclusions* $A_\alpha \longrightarrow X$ *and homomorphisms* $i_{\alpha\beta}$: $\pi_1(A_\alpha \cap A_\beta,x_0) \longrightarrow \pi_1(X,x_0)$ *induced by the inclusions* $A_\alpha \cap A_\beta \longrightarrow X$.

1. *The homomorphism*

   $\phi : \pi_1(A_1,x_0) * \pi_1(A_2,x_0) * ... * \pi_1(A_n,x_0) \longrightarrow \pi_1(X, x_0)$

   is surjective.
2. *If further each* $A_\alpha \cap A_\beta \cap A_\gamma$ *is path-connected, then the kernel*

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of ϕ is the minimal normal subgroup N generated by all elements of the form $i\alpha\beta(\omega)i\beta\alpha(\omega)-1$ for $\omega \in \pi_1(A_\alpha \cap A_\beta,x_0)$ so ϕ induces an isomorphism

$\pi_1(X,x_0)\cong$

$\pi_1(A_1,x_0) * ... * \pi_1(A_n,x_0)/N$.

Example 2.5. *Let us use Van Kampen's Theorem to show that the fundamental group of the n-sphere S^n is trivial: $\pi_1(S^n) = 0$, for all $n \geq 2$. Since S^n is path-connected, its fundamental group does not depend on the basepoint. Choose $p = (0,...,0,1)$ and $q = (0,0,...,-1)$ from R^{n+1} and consider $U = S^n -$ $\{p\}$ and $V = S^n -$ $\{q\}$. We know that $S^n = U \cup V$ is the union of open sets and $U \cap V$ is path connected. Choose $x_0 \in$ $U \cap V$. By Van Kampen's theorem, there exists a surjective homomorphism from $\pi_1(U,x_0)*\pi_1(V,x_0)$ onto $\pi_1(S^n,x_0)$. Each open U and V being homomorphic to R^n (for $n \geq 2$) are contractible. Hence $\pi_1(U,x_0) = 0$ and $\pi_1(V,x_0) = 0$. Then, their free product is a trivial group and so $\pi_1(S^n) = 0$ for $n \geq$ 2.*

Lemma 2.1. *Let $x\in$ $\mathbb{R}^n-\{0\}$. Then $\pi_1(\mathbb{R}^n-0, x) =\mathbb{Z}$ if $n = 2$* and

$\pi_1(\mathbb{R}^n-0,x) =0$ *if $n > 2$*

Proof. Note that $\mathbb{R}^n-\{0\}$ is homeomorphic to $\mathbb{R} \times S^{n-1}$ therefore, using Proposition 2.5 and *Definition* 2.3 we see that

$\pi_1(\mathbb{R}^n - \{0\},x) \tilde{=}$

$\pi_1(\mathbb{R}\times S^{n-1}(p,q)) \tilde{=}$

$\pi_1(\mathbb{R},p) \times \pi_1(S^{n-1},q)$

for some $p \in \mathbb{R}$, $q \in S^{n-1}$. Therefore, since $\pi_1(\mathbb{R},p) = 0$, we obtain

$\pi_1(\mathbb{R}^n- \{0\}) \tilde{=} \pi 1(S^{n-1},x')$

The claim then follows from our computation of the *fundamental* group of S^1 by Theorem 2.2 and Example 2.5 which gives the fundamental group of S^n, for $n \geq 2$.

III. CONTRIBUTION AND SOLUTIONS TO PROBLEMS

We will use some main results of preliminaries to answer the problems mentioned above.

A. Solution of Problem 1

To answer the Problem 1, we prove the following theorem.

Theorem 3.1. *Le G is a topological group space and let $L(G,e)$ be the loops set based on e. The fundamental group of G noted $\pi_1(G,e)$ is abelian.*

Proof. We can verify that the map

$L(G,e) \times L(G,e) \longrightarrow$

$L(G,e)$

$(\alpha, \beta)\mapsto\alpha\beta$

permits us to make operations in the quotient space

$\pi_1(G,e) \times \pi_1(G,e) \longrightarrow \pi_1(G,e)$

For this, we remark that if α and α' (respectively β and β') are two homotopic loops, and H (respectively I) is a homotopy from α to α' (resp. from β to β'), then the map $(s,t)\longrightarrow H(s,t)I(s,t)$ is a homotopy from $\alpha\beta$ to $\alpha'\beta'$. Now, let us note *that* the constant loop equal to e. Then by the concatenation definition, for all $\alpha,\beta \in L(G,e)$, we have the equality

$$(\alpha*e)(e*\beta) = \alpha*\beta. \qquad (3.1)$$

Likewise, we have

$$(e*\alpha)(\beta*e) = \beta*\alpha \qquad (3.2)$$

Moreover, since $\alpha * e$ and $e * \alpha$ are homotopic to α, and $\beta * e$ and $e * \beta$ are homotopic to β, *we* obtain that $\alpha\beta$ is homotopic to both $\alpha * \beta$ and $\beta*\alpha$. We conclude the the above map $\pi_1(G,e) \times \pi_1(G,e) \longrightarrow \pi_1(G,e)$ coincides with the usual group law, and this makes this group law commutative.

For this, we remark that if α and α^0 (respectively β and β^0) are two homotopic loops, and H (respectively I) is a homotopy from α to α^0 (resp. from β to β^0), then the map $(s,t)\mapsto H(s,t)I(s,t)$ is a homotopy from $\alpha\beta$ to $\alpha^0\beta^0$. Now, let us note *that* the constant loop equal to e. Then by the concatenation definition, for all $\alpha,\beta \in L(G,e)$, we have the equality

$$(\alpha*e)(e*\beta) = \alpha*\beta. \qquad (3.1)$$

Likewise, we have

$$(e*\alpha)(\beta*e) = \beta*\alpha \qquad (3.2)$$

Moreover, since $\alpha * e$ and $e * \alpha$ are homotopic to α, and $\beta * e$ and $e * \beta$ are homotopic to β, we obtain that $\alpha\beta$ is homotopic to both $\alpha * \beta$ and $\beta*\alpha$. We conclude the the above map $\pi_1(G,e) \times \pi_1(G,e) - \longrightarrow \pi_1(G,e)$ coincides with the usual group law, and this makes this group law commutative.

B. Solution of Problem 2

Let $p_1,...,p_n$ be n distinct points in $\mathbb{R}^2$. Let us compute

$$\pi_1(\mathbb{R}^2 - \{p_1,...,p_n\}). \qquad (3.3)$$

We are free to choose the base point x_0. Let us choose x_0 so that it does not lie on any line between two points in $\{p_1,...,p_n\}$. Let $\tilde{r}_j$ be the ray starting at x_0 and going through p_j; it follows from our choice of x_0 that the n rays $\tilde{r}_1,...,\tilde{r}_n$ are distinct. After renaming the points $p_1,...,p_n$ we may assume that the rays $\tilde{r}_1,...,\tilde{r}_n$ are ordered in a positive direction. Choose now rays $r_1,..., r_n$ with start point x_0 such that r_1 lies between $\overline{r}_n$ and $\overline{r}_1$ and r_j for $j \in \{2,...,n\}$ lies between $\overline{r}_{j-1}$ and $\overline{r}_j$. Let $\overline{A}_n$ be the infinite open wedge between r_1 and r_n containing $\overline{r}_n - \{x_0\}$; similarly for $j \in \{1,...,n-1\}$ let $\overline{A}_{jj}$ be the infinite open wedge between r_j and r_{j+1} containing $\overline{r}_j$. Take $\epsilon > 0$ small. Note that should be smaller than the distance between x_0 and r_j for each j. Let A_j be the open neighborhood of $\overline{A}_j$ (this means that we consider the set of points in $\mathbb{R}^2$ which have distance < to some point in A_j), but with the point p_j removed.

We have now the following situation:

- The objects $A_1,...,A_n$ are open and path connected subsets of $X = \mathbb{R}^2 - \{p_1,...,p_n\}$
- We have $X = \bigcup_{j=1}^{n} A_j$ and $Aj \cap Ak$ is path-connected for all $j,k \in\{1,...,n\}$

Hence Seifert VanKampen's Theorem can be applied with $A_1,...,$ A_n, in particular, the natural homomorphism

$\phi : \pi_1(A_1,x_0) * ... * \pi_1(A_n,x_0)$
$\longrightarrow \pi_1(X,x_0)$

is surjective.

Moreover, for any $j \neq k \in \{1,...,n\}$, the set $A_j \cap A_k$ is simply connected i.e. $\pi_1(A_j \cap A_k) = \{e\}$. Hence Seifert VanKampen's Theorem implies that ϕ is an isomorphism. Moreover, each A_j is homotopy equivalent with S^1; hence $\pi_1(A_j) \simeq \mathbb{Z}$ and so we conclude that

$$\pi_1(X,x_0) \simeq \mathbb{Z} * \ldots * \mathbb{Z}$$

(n-times)

is a free group with n generators. These generators are $[\gamma_1],...,[\gamma_n]$, where γ_j is a loop that is contained in A_j and goes one time around p_j.

C. Solution of Problem 3

Let us mention some spaces that are of the same homotopy as S^1. We know that S^1 is a deformation retract of the punctured plane $\mathbb{R}^2 - \{0\}$, hence $\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$; moreover, by several examples in paragraph 2.6 the cylinder in general, the spaces $S_1 = \{x \in \mathbb{R}^2 : |x| > 1$ and $S_2 = \{x \in \mathrm{R}^2 : |x| \geq 1\}$ have the fundamental group isomorphic to the infinite cyclic group $\mathbb{Z}$. In the same paragraph, we have shown that the solid torus $S^1 \times B^2$ and the product space $S^1 \times S^2$ are isomorphic to $\mathbb{Z}$. The same is true for the Mobius band. So, all these spaces whose fundamental groups are isomorphic to $\mathbb{Z}$ are of the same homotopy as S^1.

D. Solution of Problem 4

We consider here the fundamental groups of the projective space $\mathbb{R}\mathrm{P}^{\wedge n}$ for $n \geq 2$, the torus $T = S^1 \times S^1$, the cylinder $S^1 \times \mathbb{R}$, and the n- punctured plane for $n > 1$. Let us show that these surfaces and S^2 are all topologically distinct.

We know by Problem that the fundamental group of the real projective plane is $\mathbb{Z}_2$, The torus has $\mathbb{Z} \times \mathbb{Z}$ as the fundamental group, and the 2-sphere is simply connected, so $\pi_1(S^2) = \{0\}$. The fundamental group of the n-punctured plane is a free group with n generators ($n > 1$). By Proposition 2.9 all these spaces are topologically distinct.

IV. CONCLUSION AND SUGGESTIONS

We have just provided a contribution and solutions to the problems mentioned above. Our next research topic will be the etale fundamental group of an elliptic curve and homology result.

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