Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-5 Issue-1, April 2025

# New Concepts of Congruences Modulo $\frac{1}{n}$ and Positive Integers Modulo $\frac{1}{n}$

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***Abstract:*** *The aim of this article is the creation of new Mathematical beings: The positive integers Modulo 1/n. I hope that they will be used by all the scientific community in general and the computer scientists (cryptographers) particularly. In the other hand, the paper contributes to enlarge the field of number theory and algebraic sets. In this paper, I present a new type of congruences on IN: the congruences modulo* $\frac{1}{n}$ *and The Sets of positive integers modulo* $\frac{1}{n}$ *denoted respectively* $\equiv[\frac{1}{n}]$ *and* $IN^{1}_{n}$, $n \in IN^*$

*The Work is divided into two Sections:*

*Section A: It's composed of:*

- *A fundamental Theorem (With Proof)*
- *A fundamental relation of equivalence on IN denoted* $\equiv[\frac{1}{n}]$
- *The definition of the sets* $IN^{1}_{n}$, $n \in IN^*$ *(with examples)*
- *The definition of the binary operations * and Δ on IN*
- *The definition of the binary operations* $\bar{*}$ *and* $\bar{\Delta}$ *on* $IN^{1}_{n}$, $n \in IN^*$
- *The presentation of the semi-groups (IN ; *) ; (IN ; Δ) ; (*$IN^{1}_{n}$ *;* $\bar{*}$*) ; (*$IN^{1}_{n}$ *;* $\bar{\Delta}$*)*

*Section B: It's Composed of an Appendice with:*

*I) The definition and graph of the function:* $r: IN^* \to IN^*$

$x \mapsto n_0$ *Where* $n_0$ *is the rank of x.*

*II) The definition and graph of the function:* $f: IN^* \to IN^*$

$n \mapsto \left\lfloor 100^{\frac{1}{n}} \right\rfloor$

***Keywords:*** *Congruences Modulo, Positive integers Modulo, Integer Part, Partition, Equivalence Classes, Rank, Amplitude, Semi-group*

## I. INTRODUCTION

In this paper, I present two Concepts:

1. The Congruence modulo $\frac{1}{n}$ ($n \in IN^*$) defined on IN as follows:

**Manuscript received on 16 January 2025 | First Revised Manuscript received on 26 January 2025 | Second Revised Manuscript received on 16 March 2025 | Manuscript Accepted on 15 April 2025 | Manuscript published on 30 April 2025.**

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$$\forall x, y \in IN : x \equiv y[\tfrac{1}{n}] \Leftrightarrow \exists! m \in IN : \left\lfloor x^{\frac{1}{n}} \right\rfloor = \left\lfloor y^{\frac{1}{n}} \right\rfloor,$$

where $\left\lfloor x^{\frac{1}{n}} \right\rfloor$ and $\left\lfloor y^{\frac{1}{n}} \right\rfloor$ are the integer Parts of respectively $x^{\frac{1}{n}}$ and $y^{\frac{1}{n}}$ It's obviously an equivalence binary operation on IN (n is a fixed non-zero positive integer)

2. The positive integer modulo $\frac{1}{n}$ denoted $\bar{m}$ for every Positive Integer m ($m \in IN$) defined as follows: $\bar{m} = \left\{ x \in IN : \left\lfloor x^{\frac{1}{n}} \right\rfloor = m \right\}$

In fact, $\bar{m}$ is an equivalence classes of the congruence modulo $\frac{1}{n}$ defined above, therefore:

$$IN = \bigcup_{m \in IN} \bar{m}$$

the Set{ $\bar{m}$ , $m \in IN$ }denoted $IN^{1}_{n}$ is called the Set of positive integers modulo $\frac{1}{n}$

Each positive integer x of $\bar{m}$ is called a component of the positive integer modulo $\frac{1}{n}$ $\bar{m}$

- The number of components of $\bar{m}$ is called the amplitude of $\bar{m}$ and denoted $\|\bar{m}\|_{\frac{1}{n}}$
- Moreover, Every positive integer x of IN has a rank $n_0$ defined as the least non-zero positive integer Such that: $\forall n \geq n_0 : x \in \bar{1}$ of $IN^{1}_{n}$

foreword:

- In this paper, I present two new concepts on the set IN of the positive integers:

The congruence modulo $\frac{1}{n}$ and the positive integers modulo $\frac{1}{n}$, for a fixed non-zero positive integer n.

- I hope that large people of the mathematics and computer science community will take bene fit reading it

- May god, almighty, reward me for this work which contributes for the progress of mathematics in general and number theory in particular

## Section A

A fundamental Theorem:

Let n be a fixed non-zero positive integer ( $n \in IN^*$) for every positive integer x there exists a unique positive integer m such that: $m^n \leq x < (m+1)^n$ ie: $\forall x \in IN^* \exists! m \in IN : m^n \leq x < (m+1)^n$

Proof:

Let's consider the set $S_n = \{u^n \leq x , u \in IN^*\}$

- $S_n \neq \phi$ because $0^n = 0$ and $0 \leq x$ $(x \leq 1)$
- $S_n$ is upper bounded by Construction $(\forall y \in S_n : y \leq x)$

Therefore, there exists in $S_n$ a unique greatest element $m^n$.

Then: $m^n \leq x < (m+1)^n$

m is the greatest positive integer which power n is less than or equal to x.

(m+1) is the Least positive integer which power is greater than x.

ie: $\forall u \leq m : u^n \leq x$ and $\forall v \leq (m+1) : x < v^n$

Remark:

$[m^n \leq x < (m+1)^n] \Leftrightarrow [m \leq x^{\frac{1}{n}} < m+1]$ and $[m \leq x^{\frac{1}{n}} < m+1] \Leftrightarrow [\left\lfloor x^{\frac{1}{n}} \right\rfloor = m]$ where $\left\lfloor x^{\frac{1}{n}} \right\rfloor$ is the integer part of $x^{\frac{1}{n}}$

According to the remark, the fundamental theorem can be written as follows:

Let n be a fixed non-zero positive integer ( $n \in IN^*$) for every positive integer x, there exists a unique positive integer m such that $\left\lfloor x^{\frac{1}{n}} \right\rfloor$ (for n=1, $x^{\frac{1}{1}} = x; \lfloor x \rfloor = x; x^1 \leq x < (x+1)^1$)

Special case: **n=2**. if n=2 we can state for every positive integer x, there exists a unique positive integer m such that $m^2 \leq x < (m+1)^2$ or $m \leq x^{\frac{1}{2}} < m+1$ ie: $\left\lfloor x^{\frac{1}{2}} \right\rfloor = m$

Note that: $[m^2 \leq x < (m+1)^2] \Leftrightarrow [m \leq x^{\frac{1}{2}} < m+1] \Leftrightarrow \left[\left\lfloor x^{\frac{1}{2}} \right\rfloor = m\right]$

A fundamental equivalence relation on IN:

Let's consider the relation denoted $\equiv \left\lfloor \frac{1}{n} \right\rfloor$ defined on IN by: $U \equiv V \left\lfloor \frac{1}{n} \right\rfloor \Leftrightarrow \exists! m \in IN : \left\lfloor U^{\frac{1}{n}} \right\rfloor = \left\lfloor V^{\frac{1}{n}} \right\rfloor$ Where n is a fixed non-zero positive integer($n \in IN^*$).

This relation is by definition on equivalence relation which equivalence classes $\overline{m}$ are defined by:

$$\overline{m} = \left\{ x \in IN : \left\lfloor x^{\frac{1}{n}} \right\rfloor = m \right\}$$

The classes $\overline{m}$ /$m \in IN$ form a partition of IN

Each classes $\overline{m}$ is composed of a finite number of positive Integers U:

$$\left\lfloor U^{\frac{1}{n}} \right\rfloor = m : \text{ie } \exists! k \in IN^* : card(\overline{m}) = k$$

Terminology: Let's call:

- ) $\equiv$ the congruence module $\frac{1}{n}$ ( $\frac{1}{n}$ the module of $\equiv$)
- ) the Set $\overline{m} = \left\{ x \in IN : \left\lfloor x^{\frac{1}{n}} \right\rfloor = m \right\}$ the integer m modulo $\frac{1}{n}$
- ) x is a component of $\overline{m}$
- ) $\{\overline{m}\}_{m \in IN}$ Will be denoted $IN_n^1$ for a fixed non-zero positive integer n ($n \in IN^*$).

$IN_n^1 = \{\overline{0}, \overline{1}, \overline{2}, \ldots\ldots\ldots, \overline{m}, \ldots\ldots\}$

Let's examine the case n = 2:

Retrieval Number:100.1/ijam.A118805010425
DOI: 10.54105/ijam.A1188.05010425
Journal Website: www.ijam.latticescipub.com



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the congruence modulo $\frac{1}{n}$ is defined as follows: $\forall U, V \in IN : U \equiv V\left[\frac{1}{2}\right] \Leftrightarrow \exists! m \in IN : \left\lfloor U^{\frac{1}{2}} \right\rfloor = \left\lfloor V^{\frac{1}{2}} \right\rfloor = m$

$\overline{m} = \left\{x \in IN : \left\lfloor x^{\frac{1}{2}} \right\rfloor = m\right\}$

Examples:

$\overline{0} = \{ x \in IN : \left\lfloor x^{\frac{1}{2}} \right\rfloor = 0 \} = \{0\}$

0 is the unique component of $\overline{0}$

$\overline{1} = \{ x \in IN : \left\lfloor x^{\frac{1}{2}} \right\rfloor = 1 \} = \{1;2;3\}$

1;2;3 are the three components of $\overline{1}$

$\overline{2} = \{ x \in IN : \left\lfloor x^{\frac{1}{2}} \right\rfloor = 2 \} = \{4;5;6;7;8\}$

4;5;6;7;8 are the five components of $\overline{2}$

$IN_2^1 = \{ \overline{0}; \overline{1}; \overline{2}; \ldots\ldots\ldots \overline{m}; \ldots\ldots \}$

Each element of $IN_2^1$ is a positive integer modulo $\frac{1}{2}$ is infinite

Each positive integer modulo $\frac{1}{2}$ $\overline{m}$ has a finite number of components equal to the number of positive integers of the interval [m², $(m+1)^2$[= 2m + 1 because card ([m², $(m+1)^2$[) =2m+1.

Now here are some examples of positive integers modulo $\frac{1}{2}$ $\overline{m}$ and their 2m+ 1 components

$\overline{0}$= {0} (1 component: 1 = 2(0)+1)
$\overline{1}$= {1;2;3} (3 components; 3=2(1)+1)
$\overline{2}$= {4;5;6;7;8} (5 components, 5=2(2)+1)
$\overline{3}$= {9;10;11;12;13;14;15} (7 components; 7=2(3)+1)
$\overline{4}$= {6;17;18;19;20,21,24;23;24} (19 components 9 =2(4)+1)

Each component x of $\overline{m}$ satisfy: m² ≤ x <(m+1)² or: $\left\lfloor x^{\frac{1}{2}} \right\rfloor = m$

Note that $IN_2^1 = \{ \overline{0}; \overline{1}; \overline{2}; \ldots\ldots\ldots \overline{m} \}$ where $\overline{m} = \{x \in IN: m \le x < m + 1\}$

Then: $\overline{0}$={0} ; $\overline{1}$={1} ; $\overline{2}$={2} ; ……….. ; $\overline{m}$={m}

The relation ≡ is the equality on IN ie $U \equiv V\left[\frac{1}{1}\right] \Leftrightarrow U = V$

Proposition (1): given $n \in IN^*$

for every non-zero positive integer x there exists a unique non-zero positive integer $n_0$ such that: x is a component of $\overline{1}$ of $IN_n^1$ for all values of n greater than or equal to n: ie: $\forall n \ge n_0 : x \in \overline{1}$ of $IN_n^1$

Proof:

The proof of this proposition calls upon the Following property:

**For Every Non-Zero Positive Integer X There Exists A Unique Non-Zero Positive Integer $N_0$ Such That: $x < 2^{n_0}$**

Proof: Let's consider the set $S_x = \{ 2^U : 2^U \le x : U \in IN \}$

- ) $S_x \neq \emptyset$ because $2^0=1<x$ (x≥1)
- ) $S_x$ is upper bounded by construction ( $\forall y \in S_x : y \le x$ ), therefore there exists a unique greatest element of $S_x$ ie; there exists a unique non-zero integer m Such that, $2^m \le x$ and for all k<m we have $2^k<x$

That means that $2^{m+1}$ is greater than x ie: $1 \le x < 2^{m+1}$

Let's put $m+1=n_0$; then $m=n_0-1$

$2^{n_0}$ is the least power of 2 greater than x.

$2^{n_0-1}$ is the greatest power of 2 least than or equal to x

Retrieval Number:100.1/ijam.A118805010425
DOI: 10.54105/ijam.A1188.05010425
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ie: $\forall n \geq n_0 : 1 \leq x < 2^n$

$$\left(1 \leq x < 2^n\right) \Leftrightarrow \left(1 \leq x^{\frac{1}{n}} < 2\right) \Leftrightarrow \left(\left\lfloor x^{\frac{1}{n}} \right\rfloor = 1\right) \Leftrightarrow \left(x \in \bar{1} \text{ of } IN_n^1\right)$$

Note that x = 0 is the unique component of $\bar{0}$ of $IN_n^1$ $(n \in IN^*)$ because $\forall n \in IN^* : \left\lfloor 0^{\frac{1}{n}} \right\rfloor = 0^{\frac{1}{n}} = 0$

ie: $\forall n \in IN^* : 0 \in \bar{0} \text{ of } IN_n^1$

Examples:

1) x = 10: $2^3 < 10 < 2^4$, $\forall n \geq 4 : 10 < 2^n$, $n_0 = 4$, $\forall n \geq 4 : 10 \in \bar{1}$ of $IN_n^1$
2) x = 100: $2^6 < 100 < 2^7$, $\forall n \geq 7 : 100 < 2^n$, $n_0 = 7$, $\forall n \geq 7 : 10 \in \bar{1}$ of $IN_n^1$
3) x = 1000: $2^9 < 1000 < 2^{10}$, $\forall n \geq 10 : 1000 < 2^n$, $n_0 = 10$, $\forall n \geq 10 : 1000 \in \bar{1}$ of $IN_n^1$

Rank of a non-zero positive integer:

According to the property: $\forall x \in IN^*, \exists! n_0 \in IN^* : \forall n \geq n_0 : x < 2^n$

Let's call $n_o$ the rank of the non-zero positive integer x: $n_o = r(x)$.

$\forall n \geq n_0 : x \in \bar{1}$ of $IN_n^1$

Remark: $\forall n \in IN^* : 0^{\frac{1}{n}} = \left\lfloor 0^{\frac{1}{n}} \right\rfloor = 0$

therefore $\forall n \in IN^* : 0 \in \bar{0}$ of $IN_n^1$, 0 has no rank (r(0) does not exist)

Finally, the rank $n_0$ of the non-zero positive integer x

($x \in IN^*$) is the least non-zero positive integer $n_0$

satisfying: $2^{n_0 - 1} \leq x < 2^{n_0}$ note that the interval $[2^{n_0-1}; 2^{n_0}[$ of $IN^*$ can contain more than one element: they have all the same rank $n_0$. ie: $\forall x : 2^{n_0-1} \leq x < 2^{n_0}$. $r(x) = n_0$

Examples :

- $[2^0, 2^1[ = [1,2[ = \{1\}$
  $n_0 = r(1) = 1$
- $[2^1, 2^2[ = [2,4[ = \{2,3\}$
  $n_0 = r(2) = r(3) = 2$
- $[2^2, 2^3[ = [4,8[ = \{4,5,6,7\}$
  $n_0 = 3 = r(4) = r(5) = r(6) = r(7)$
- $[2^3, 2^4[ = [8,16[ = \{8,9,10,11,12,13,14,15\}$

$n_0 = r(8) = r(9) = r(10) = r(11) = r(12) = r(13) = r(14) = r(15) = 4$

- The intervals [2i 2i+1[/ i∈IN forma partition of IN*

Proposition (02):

For every non-zero positive integer x there exists k positive integers modulo $\frac{1}{n}$ $\bar{m}$ distinct of $\bar{1}$ where x is one of their components (K< $n_0$).

Meaning that x is one of the components of $\bar{1}$ of $IN_n^1$ for all n : $n \geq n_0$.

Proof :

- We know that for every non-zero positive integer x there exists a unique non-zero positive Integer $n_0$ such that: $\forall n \geq n_0 : x \in \bar{1}$ of $IN_n^1$

In the other hand, the function $f : \{1;2;3;4;\ldots\ldots;n_0 - 1\} \to IN^*$.

$n \mapsto \left\lfloor x^{\frac{1}{n}} \right\rfloor$

for a fixed Value of the non-zero positive integer x : $x \geq 2$is a decreasing function,
therefore : for all non-zero positive integers n such that $n < n_0$we have $\left\lfloor x^{\frac{1}{n}} \right\rfloor > \left\lfloor x^{\frac{1}{n_0}} \right\rfloor$ since $\left\lfloor x^{\frac{1}{n_0}} \right\rfloor = 1$

then $\forall n < n_0 : \left\lfloor x^{\frac{1}{n}} \right\rfloor > 1$

Card {1 ;2 ;3 ;…. ;$n_0$-1}=$n_0$-1, therefore there exists at most $n_0$-1 positive integers modulo $\frac{1}{n}$, $\overline{m}$ distinct of $\bar{1}$ where x is one of their components

if x = 1 $\forall n \geq 1 : 1^n \leq 1 < 2^n \Leftrightarrow 1 \leq 1^{\frac{1}{n}} < 2$

$\left\lfloor 1^{\frac{1}{n}} \right\rfloor = 1$

$x \in \bar{1}$ of $IN_n^1$

K=0 = $n_0$ – 1 ($n_0$=1)

<u>Examples:</u>

| x = 10 | | |
|---|---|---|
| n = 1 | n = 2 | n = 3 |
| $10^1 \leq 10 < 11^1$<br>$10 \leq 10^{\frac{1}{1}} < 11$<br>$[10^{\frac{1}{1}}] = 10$<br>$x \in \overline{10}$ of $IN_1^1$ | $3^2 \leq 10 < 4^2$<br>$3 \leq 10^{\frac{1}{2}} < 4$<br>$[10^{\frac{1}{2}}] = 3$<br>$x \in \bar{3}$ of $IN_2^1$ | $2^3 \leq 10 < 3^3$<br>$2 \leq 10^{\frac{1}{3}} < 3$<br>$[10^{\frac{1}{3}}] = 2$<br>$x \in \bar{2}$ of $IN_3^1$ |
| $\forall n \geq 4 : 1^n < 10 < 2^n$ then $1 \leq 10^{\frac{1}{n}} < 2$ then<br>$[10^{\frac{1}{n}}] = 1$ | | |

| x = 100 | | | | | |
|---|---|---|---|---|---|
| n = 1 | n = 2 | n = 3 | n = 4 | n = 5 | n = 6 |
| $100^1 \leq 100 < 101^1$<br>$100 \leq 100^{\frac{1}{1}} < 101$<br>$[100^{\frac{1}{1}}] = 100$<br>$x \in \overline{100}$ of $IN_1^1$ | $10^2 \leq 100 < 11^2$<br>$10 \leq 10^{\frac{1}{2}} < 11$<br>$[10^{\frac{1}{2}}] = 10$<br>$x \in \overline{10}$ of $IN_2^1$ | $4^3 \leq 100 < 5^3$<br>$4 \leq 10^{\frac{1}{3}} < 5$<br>$[10^{\frac{1}{3}}] = 4$<br>$x \in \bar{4}$ of $IN_3^1$ | $3^4 < 100 < 4^4$<br>$3 < 100^{\frac{1}{4}} < 4$<br>$[100^{\frac{1}{4}}] = 3$<br>$x \in \bar{3}$ of $IN_4^1$ | $2^5 < 100 < 3^5$<br>$2 < 100^{\frac{1}{5}} < 3$<br>$[100^{\frac{1}{5}}] = 2$<br>$x \in \bar{2}$ of $IN_5^1$ | $2^6 < 100 < 3^6$<br>$2 < 100^{\frac{1}{6}} < 3$<br>$[100^{\frac{1}{6}}] = 2$<br>$x \in \bar{2}$ of $IN_6^1$ |
| Since: $2^{10} > 1000$ then $\forall n \geq 7 : 1^n < 100 < 2^n$ then $1 < 1000^{\frac{1}{n}} < 2$ then $[1000^{\frac{1}{n}}] = 1$<br>$x \in \bar{1}$ of $IN_n^1$ ; $n_0$ = 7; K = 5 < $n_0$ – 1 = 7 – 1 | | | | | |

| x = 1000 | | | | |
|---|---|---|---|---|
| n = 1 | n = 2 | n = 3 | n = 4 | n = 5 |
| $1000^1 < 1000 < 1001^1$<br>$1000 < 1000^{\frac{1}{1}} < 1001$<br>$[1000^{\frac{1}{1}}] = 1000$<br>$x \in \overline{1000}$ of $IN_1^1$ | $31^2 < 1000 < 32^2$<br>$31 < 1000^{\frac{1}{2}} < 32$<br>$[1000^{\frac{1}{2}}] = 31$<br>$x \in \overline{31}$ of $IN_2^1$ | $10^3 < 1000 < 11^3$<br>$10 < 1000^{\frac{1}{3}} < 11$<br>$[1000^{\frac{1}{3}}] = 10$<br>$x \in \overline{10}$ of $IN_3^1$ | $5^4 < 1000 < 6^4$<br>$5 < 1000^{\frac{1}{4}} < 6$<br>$[1000^{\frac{1}{4}}] = 5$<br>$x \in \bar{5}$ of $IN_4^1$ | $3^5 < 1000 < 4^5$<br>$3 < 1000^{\frac{1}{5}} < 4$<br>$[1000^{\frac{1}{5}}] = 3$<br>$x \in \bar{3}$ of $IN_5^1$ |

| | | | | |
|---|---|---|---|---|
| | | | | |
| n = 6 | n = 7 | n = 8 | n = 9 | |
| $3^6 < 1000 < 4^6$<br>$3 < 1000^{\frac{1}{6}} < 4$<br>$[1000^{\frac{1}{6}}] = 3$<br>$x \in \bar{3}$ of $IN_6^1$ | $2^7 < 1000 < 3^7$<br>$2 < 1000^{\frac{1}{7}} < 3$<br>$[1000^{\frac{1}{7}}] = 2$<br>$x \in \bar{2}$ of $IN_7^1$ | $2^8 < 1000 < 3^8$<br>$2 < 1000^{\frac{1}{8}} < 3$<br>$[1000^{\frac{1}{8}}] = 2$<br>$x \in \bar{2}$ of $IN_8^1$ | $2^9 < 1000 < 3^9$<br>$2 < 1000^{\frac{1}{9}} < 3$<br>$[1000^{\frac{1}{9}}] = 2$<br>$x \in \bar{2}$ of $IN_9^1$ | |
| Since: $2^{10} > 1000$ then $\forall n \geq 10 : 1^n < 1000 < 2^n$ then $1 < 1000^{\frac{1}{n}} < 2$ then $[1000^{\frac{1}{n}}] = 1$<br>$x \in \bar{1}$ of $IN_n^1$ ; $n_0 = 10$; $K = 6 < n_0 - 1 = 10 - 1$ | | | | |

Amplitude of a positive integer modulo $\frac{1}{n}$, $n \in IN^*$

Let's remind that:

$$\forall m \in IN : \bar{m} = \left\{ x \in IN : [x^{\frac{1}{n}}] = m \right\} = \left\{ x \in IN : m^n \leq x \leq (m+1) \right\}$$

x is called a component of $\bar{m}$

**Definition and notation:**

The number of components of the positive integer modulo $\frac{1}{n}$,

$\bar{m}$ is called the amplitude of $\bar{m}$ and denoted: $\|\bar{m}\|_{\frac{1}{n}}$.

**Examples:**

For n = 1

$\|\bar{m}\|_{\frac{1}{1}} = 1$ because $\bar{m} = \{m\}$. m is the unique component of $\bar{m}$

For n = 2

$\bar{m} = \{x \in IN: \lfloor x^{\frac{1}{2}} \rfloor = m\} = \{x \in IN: m^2 \leq x < (m+1)^2\}$

Card($\bar{m}$) = Card$[m^2 ; (m+1)^2[$ = 2m + 1 , then $\|\bar{m}\|_{\frac{1}{2}}$ = 2m + 1

let's calculate $\|\bar{1}\|_{\frac{1}{3}}$, $\|\bar{1}\|_{\frac{1}{4}}$

- in $IN_3^1$, $\bar{1} = \{x \in IN: \lfloor x^{\frac{1}{3}} \rfloor = 1\} = \{x \in IN: 1^2 \leq x < 2^2\} = \{1 ; 2 ; 3 ; 4 ; 5 ; 6 ; 7\}$ ; Card($\bar{1}$) = 7

Therefore $\|\bar{1}\|_{\frac{1}{3}}$ = 7

- in $IN_4^1$, $\bar{1} = \{x \in IN: \lfloor x^{\frac{1}{4}} \rfloor = 1\} = \{x \in IN: 1^4 \leq x < 2^4\} = \{1 ; 2 ; 3 ; 4 ; 5 ;………; 15\}$ ; Card($\bar{1}$) = 15

Therefore $\|\bar{1}\|_{\frac{1}{4}}$ = 15

**Question:** How can we define a binary operation on $IN_n^1$ ?

**Answer:** To define a binary operation on $IN_n^1$ compatible with the congruence modulo $\frac{1}{n}$ ($\equiv$)

That means that * and $\equiv$ must satisfy:

$\forall$ x ; y ; x' ; y' $\in$ IN if :( $x \equiv y[\frac{1}{n}]$ and $x' \equiv y'[\frac{1}{n}]$) then ($(x * x') \equiv (y * y')[\frac{1}{n}]$)





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This compatibility allows us to define the binary operation $\bar{*}$ on $IN_n^1$ as follows:

$$\forall\, x ; y \in IN: \bar{x} \bar{*} \bar{y} = \overline{(x * y)}$$

In that case;

- if * is commutative on IN, then $\bar{*}$ is commutative on $IN_n^1$
- if * is associative on IN, then $\bar{*}$ is associative on $IN_n^1$
- if e is the unit element of * on IN, then $\bar{e}$ is the unit element of $\bar{*}$ on $IN_n^1$
- if 0 is the zero element of * on IN, then $\bar{0}$ is the zero element of $\bar{*}$ on $IN_n^1$
- if $x'$ is the inverse element of x on IN (ie x * $x'$ = $x'$ * x = e), then $\overline{x'}$ is the inverse element of $\bar{x}$ on $IN_n^1$ ( ie $\overline{x'} \bar{*} \bar{x} = \bar{x} \bar{*} \overline{x'} = \bar{e}$)

so: all the problem is to find a binary operation on IN that is compatible with the congruence modulo $\frac{1}{n}$

**Two Examples of Binary Opérations on IN Compatible with the Congruence Modulo $\frac{1}{n}$**

**1)** Let's consider on IN the binary operations * defined as follows;

$$\forall\, x; y \in IN: x * y = \left\lfloor x^{\frac{1}{n}} \right\rfloor + \left\lfloor y^{\frac{1}{n}} \right\rfloor$$

where $\left\lfloor x^{\frac{1}{n}} \right\rfloor$ and $\left\lfloor y^{\frac{1}{n}} \right\rfloor$ are respectively the integer part of $x^{\frac{1}{n}}$ and $y^{\frac{1}{n}}$

this operation is compatible with the congruence modulo $\frac{1}{n}$ because:

U ; V ; $U'$ ; $V' \in$IN, if : $U \equiv U'\left[\frac{1}{n}\right]$ and $V \equiv V'\left[\frac{1}{n}\right]$then $U * V \equiv U' * V'\left[\frac{1}{n}\right]$

<u>Proof:</u>

U * V = $\left\lfloor U^{\frac{1}{n}} \right\rfloor + \left\lfloor V^{\frac{1}{n}} \right\rfloor$; since $U \equiv U'\left[\frac{1}{n}\right]$ and $V \equiv V'\left[\frac{1}{n}\right]$

means that: $\left\lfloor U^{\frac{1}{n}} \right\rfloor = \left\lfloor U'^{\frac{1}{n}} \right\rfloor$ and $\left\lfloor V^{\frac{1}{n}} \right\rfloor = \left\lfloor V'^{\frac{1}{n}} \right\rfloor$

therefore U * V = $\left\lfloor U^{\frac{1}{n}} \right\rfloor + \left\lfloor V^{\frac{1}{n}} \right\rfloor = \left\lfloor U'^{\frac{1}{n}} \right\rfloor + \left\lfloor V'^{\frac{1}{n}} \right\rfloor = U' + V'$

ie: (U*V) = $(U' * V')$ thus: $(U*V) \equiv (U' * V')$

* is then compatible with the congruence modulo $\frac{1}{n}$

**2)** Similarly: Let's consider the binary operation Δ defined on IN as follows:

$$\forall\, x; y; x'; y' \in IN: x \,\Delta\, y = \left\lfloor x^{\frac{1}{n}} \right\rfloor \times \left\lfloor y^{\frac{1}{n}} \right\rfloor$$

As for *, Δ is compatible with the congruence modulo $\frac{1}{n}$ on IN

ie: U ; V ; $U'$ ; $V' \in$IN , if : $U \equiv U'\left[\frac{1}{n}\right]$ and $V \equiv V'\left[\frac{1}{n}\right]$ then $U \Delta V \equiv U' \Delta V'\left[\frac{1}{n}\right]$

Note that both * and Δ are associative and commutative.

<u>Consequence:</u> construction of two binary operation associative and commutative on $IN_n^1$

Since * and Δ are two binary operations associative and commutative on IN compatible with the congruence modulo $\frac{1}{n}$ $\bar{*}$ and $\bar{\Delta}$ defined on $IN_n^1$ as follows:

$\forall m; m' \in IN : \bar{m} \bar{*} \overline{m'} = \overline{(m * m')}$ and $\forall m; m' \in IN : \bar{m} \bar{\Delta} \overline{m'} = \overline{(m \Delta m')}$

Are two binary operations associative and commutative on $IN_n^1$

Remark: both $\Delta$ and $\overline{\Delta}$ are distributive with respect to respectively $*$ and $\overline{*}$ on respectively IN and $IN_n^1$

$\forall$ x ; y ; z $\in$ IN: x $\Delta$ (y * z) = (x $\Delta$ y) * (x $\Delta$ y) and (y $\Delta$ z) * x = (y * x) $\Delta$ (z * x)

and

$\forall$ U ; V ; W $\in$ IN: $\overline{U}\,\overline{\Delta}(\overline{V}\,\overline{*}\,\overline{W}) = (\overline{U}\,\overline{\Delta}\,\overline{V})\overline{*}(\overline{U}\,\overline{\Delta}\,\overline{W})$ and $(\overline{V}\,\overline{\Delta}\,\overline{W})\overline{*}\,\overline{U} = (\overline{V}\,\overline{*}\,\overline{U})\overline{\Delta}(\overline{W}\,\overline{*}\,\overline{U})$

The semi-groups (IN; *) , (IN ; $\Delta$) , $\left(IN_n^1;\overline{*}\right)$ ; $\left(IN_n^1;\overline{\Delta}\right)$

both * and $\Delta$ are associative and commutative

- neither * nore $\Delta$ have a unit element
- *has no zero element
- $\Delta$ have 0 as zero element because: $\forall$ x $\in$ IN: x $\Delta$ 0 $= \left\lfloor x^{\frac{1}{n}} \right\rfloor \times \left\lfloor 0^{\frac{1}{n}} \right\rfloor = \left\lfloor x^{\frac{1}{n}} \right\rfloor \times 0 = 0$

so: (IN ; *) is a commutative semi-group without unit element and without zero element

(IN ; $\Delta$) is a commutative semi-group with 0 as zero element and without unit element

both $\overline{*}$ and $\overline{\Delta}$ are associative and commutative neither $\overline{*}$ nore $\overline{\Delta}$ have a unit element

$\overline{\Delta}$ has $\overline{0}$ as zero element because: $\forall\ \overline{U} \in IN_n^1 :\ \overline{U}\,\overline{\Delta}\,\overline{0} = \overline{0}$

so: $\left(IN_n^1;\overline{*}\right)$ is a commutative semi-group without a unit element and without a zero element

$\left(IN_n^1;\overline{\Delta}\right)$ is a commutative semi-group with $\overline{0}$ as a zero element and without a unit element

two subsets of (IN; $\Delta$) and $\left(IN_n^1;\overline{\Delta}\right)$

- Let's consider $(\{0;1\};\Delta)$ there is the table of $\Delta$:

| $\Delta$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

as shown on the table:

0 is the zero element

1 is the unit element

moreover $\Delta$ is commutative and associative so: ({0; 1}; $\Delta$) is a commutative semi-group with zero element 0 and with unit ellement 1

- Let's consider $(\{\overline{0};\overline{1}\};\overline{\Delta})$ there is the table of $\overline{\Delta}$:

| $\overline{\Delta}$ | $\overline{0}$ | $\overline{1}$ |
|---|---|---|
| $\overline{0}$ | $\overline{0}$ | $\overline{0}$ |
| $\overline{1}$ | $\overline{0}$ | $\overline{1}$ |

As shown on the table:

$\overline{0}$ is the zero element

$\overline{1}$ is the unit element

moreover $\overline{\Delta}$ is commutative and associative so : $(\{\overline{0};\overline{1}\};\overline{\Delta})$ is a semi-group with zero element $\overline{0}$ and with unit element $\overline{1}$

note that {0 ; 1} is not closed under * because: 1*1= 2$\notin${0 ; 1}

and $\{\overline{0}\ ;\ \overline{1}\}$ is not closed under $\overline{*}$ because: $\overline{1}\,\overline{*}\,\overline{1} = \overline{2} \notin \{\overline{0}\ ;\ \overline{1}\}$

Definition of both * and $\Delta$ on IN for n = 2 and both $\overline{*}$ and $\overline{\Delta}$ on $IN_n^1$

Retrieval Number:100.1/ijam.A118805010425
DOI: 10.54105/ijam.A1188.05010425
Journal Website: www.ijam.latticescipub.com



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$$\forall\, x\,;\, y \in \text{IN}:\ x * y = \left\lfloor x^{\frac{1}{2}} \right\rfloor + \left\lfloor y^{\frac{1}{2}} \right\rfloor \quad \text{and} \quad x\,\Delta\, y = \left\lfloor x^{\frac{1}{2}} \right\rfloor \times \left\lfloor y^{\frac{1}{2}} \right\rfloor$$

$$\forall\, U\,;\, V \in \text{IN}:\ \bar{U}\ \bar{*}\ \bar{V} = \overline{U * V} = \overline{\left\lfloor U^{\frac{1}{2}} \right\rfloor + \left\lfloor V^{\frac{1}{2}} \right\rfloor} \ \text{and}\ \bar{U}\ \bar{\Delta}\ \bar{V} = \overline{U \Delta V} = \overline{\left\lfloor U^{\frac{1}{2}} \right\rfloor \times \left\lfloor V^{\frac{1}{2}} \right\rfloor}$$

- numerical <u>examples:</u>

Here are some numerical examples to clarify the situation :

| | |
|---|---|
| ➢ $\overline{1*3} = \left\lfloor 1^{\frac{1}{2}} \right\rfloor + \left\lfloor 3^{\frac{1}{2}} \right\rfloor = 1 + 1 = 2$ | ➢ $\overline{1\Delta 3} = \left\lfloor 1^{\frac{1}{2}} \right\rfloor + \left\lfloor 3^{\frac{1}{2}} \right\rfloor = 1 \times 1 = 1$ |
| ➢ $\overline{5*2} = \left\lfloor 5^{\frac{1}{2}} \right\rfloor + \left\lfloor 2^{\frac{1}{2}} \right\rfloor = 2 + 1 = 3$ | ➢ $\overline{5\Delta 2} = \left\lfloor 5^{\frac{1}{2}} \right\rfloor \times \left\lfloor 2^{\frac{1}{2}} \right\rfloor = 2 \times 1 = 2$ |
| ➢ $\overline{7*10} = \left\lfloor 7^{\frac{1}{2}} \right\rfloor + \left\lfloor 10^{\frac{1}{2}} \right\rfloor = 2 + 3 = 5$ | ➢ $\overline{7\Delta 10} = \left\lfloor 7^{\frac{1}{2}} \right\rfloor + \left\lfloor 10^{\frac{1}{2}} \right\rfloor = 2 \times 3 = 6$ |

**Therefore:**

| | |
|---|---|
| ➢ $\bar{1}\ \bar{*}\ \bar{3} = \overline{1*3} = \bar{2}$ | ➢ $\bar{1}\ \bar{\Delta}\ \bar{3} = \overline{1\Delta 3} = \bar{1}$ |
| ➢ $\bar{5}\ \bar{*}\ \bar{2} = \overline{5*2} = \bar{3}$ | ➢ $\bar{5}\ \bar{\Delta}\ \bar{2} = \overline{5\Delta 2} = \bar{2}$ |
| ➢ $\bar{7}\ \bar{*}\ \overline{10} = \overline{7*10} = \bar{5}$ | ➢ $\bar{7}\ \bar{\Delta}\ \overline{10} = \overline{7\Delta 10} = \bar{6}$ |

**<u>Final Notation;</u>**

To make it easy for the readers, let's replace : * , Δ on IN and $\bar{*}, \bar{\Delta}$ on $\mathbb{N}_n^1$ by respectively + and × on both IN and $\mathbb{N}_n^1$.

ie : $\forall x, y \in \text{IN} : x * y = x + y$, and xΔy= x×y

$\forall U, V \in \text{IN} : \bar{U}\ \bar{*}\ \bar{V} = \bar{U} + \bar{V}$ and $\bar{U}\ \bar{\Delta}\ \bar{V} = \bar{U}\ \bar{V} \times$

**<u>Consequently:</u>**

(IN ; *) ; (IN ; Δ) ; ( $\mathbb{N}_n^1$ ; $\bar{*}$) ; ( $\mathbb{N}_n^1$ ; $\bar{\Delta}$ ) ; ({0 ;1} ; Δ) ; ({$\bar{0}$ ;$\bar{1}$} ; $\bar{\Delta}$ ) will be dended :

(IN ; +) ; (IN ; ×) ; ( $\mathbb{N}_n^1$ ;+) ; ( $\mathbb{N}_n^1$ ;×) ; ({0 ;1} ; ×) ; ({$\bar{0}$ ;$\bar{1}$} ; ×)

{0 ;1} and {$\bar{0}$ ;$\bar{1}$} are not closed under +

## <u>Section B</u>

<u>Appendice</u>

I) <u>Let's consider the function</u>

$$r : \text{IN}^* \to \text{IN}^*$$
$$x \mapsto n_0$$

where $n_0$ is the rank of x.

($n_0$ is the least positive integer such that: $x < 2^{n_0}$ )

By construction r is an increasing function.

Here are same values of x and the corresponding values of r(x) = $n_0$

$x = 1 : 2^0 \leq 1 < 2^1 : n_0 = 1$

$x = 2 : 2^1 \leq 2 < 2^2 : n_0 = 2$

$x = 3 : 2^1 \leq 3 < 2^2 : n_0 = 2$

$x = 4 : 2^2 \leq 4 < 2^3 : n_0 = 3$

$x = 5 : 2^2 \leq 5 < 2^3 : n_0 = 3$

$x = 9 : 2^3 \leq 9 < 2^4 : n_0 = 4$

$x = 10 : 2^3 \leq 10 < 2^4 : n_0 = 4$

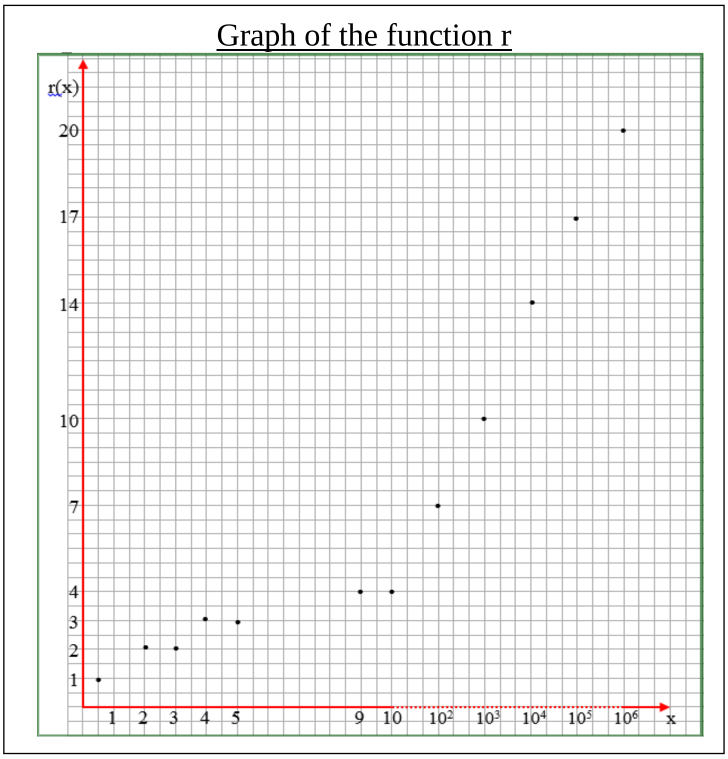
$x = 10^2 : 2^6 \leq 10^2 < 2^7 : n_0 = 7$

$x = 10^3 : 2^9 \leq 10^3 < 2^{10} : n_0 = 10$

$x = 10^4 : 2^{13} \leq 10^4 < 2^{14} : n_0 = 14$

$x = 10^5 : 2^{16} \leq 10^5 < 2^{17} : n_0 = 17$

$x = 10^6 : 2^{19} \leq 10^6 < 2^{20} : n_0 = 20$




## II. RECAPITULATION

| x = | 1 | 2 | 3 | 4 | 5 | 9 | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r(x) = | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 7 | 10 | 14 | 17 | 20 |

II) let's consider the function:f: $\mathbb{N}^* \to \mathbb{N}^*$

$$n \mapsto \left\lfloor 100^{\frac{1}{n}} \right\rfloor$$

- f \{1 ; 2 ; 3 ; 4 ; 5 ; 6} is a decreasing function.

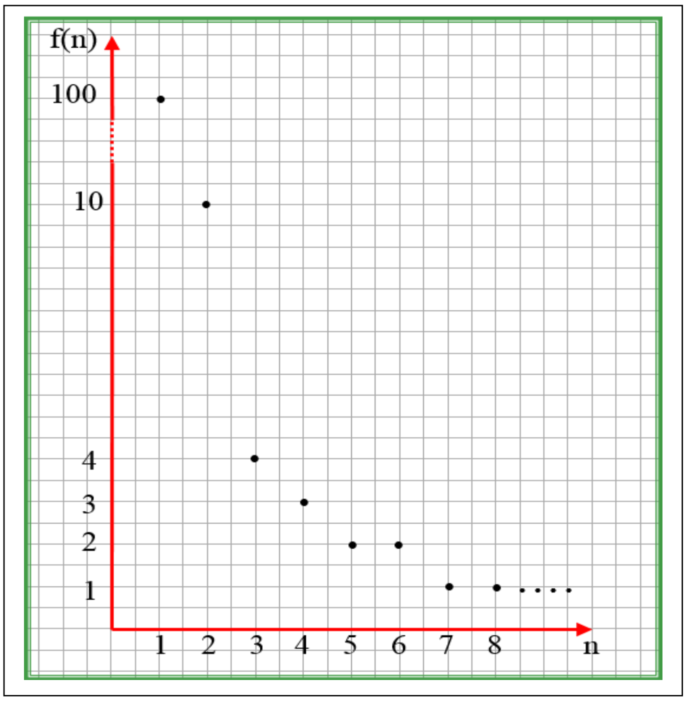
$\forall$ n $\in$ {1 ; 2 ; 3 ; 4 ; 5 ; 6} : f(n) $\neq$ 1

$\forall$ n $\geq$ 7 : f(n) = 1 because :$2^7 > 100$

graph of the function f :

| n = | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 …. |
|---|---|---|---|---|---|---|---|---|---|
| f(n) = | 100 | 10 | 4 | 3 | 2 | 2 | 1 | 1 | 1 ….. |

$\forall$ n $\geq$ 7 : f(n) = 1




explanation:

$$n = 1 : \left\lfloor 100^{\frac{1}{1}} \right\rfloor = \lfloor 100 \rfloor = 100 \ ; f(1) = 100$$

$$n = 2 : \left\lfloor 100^{\frac{1}{2}} \right\rfloor = \lfloor 10 \rfloor = 10 \ ; f(2) = 10$$

$n = 3 : 4^3 < 100 < 5^3$

$$4 < 100^{\frac{1}{3}} < 5; \left\lfloor 100^{\frac{1}{3}} \right\rfloor = 3 \ ; f(3) = 4$$

$n = 4 : 3^4 < 100 < 4^4$

Retrieval Number:100.1/ijam.A118805010425
DOI: 10.54105/ijam.A1188.05010425
Journal Website: www.ijam.latticescipub.com



Published By:
Lattice Science Publication (LSP)

$$3 < 100^{\frac{1}{4}} < 4; \left\lfloor 100^{\frac{1}{4}} \right\rfloor = 3 \text{ ; } f(4) = 3$$

n = 5 : $2^5 < 100 < 3^5$

$$2 < 100^{\frac{1}{3}} < 3; \left\lfloor 100^{\frac{1}{5}} \right\rfloor = 2 \text{ ; } f(5) = 2$$

n = 6 : $2^6 < 100 < 3^6$

$$2 < 100^{\frac{1}{6}} < 3; \left\lfloor 100^{\frac{1}{6}} \right\rfloor = 2 \text{ ; } f(6) = 2$$

$$\forall n \geq 7 : 1^n < 100 < 2^n \text{ then } 1 < 100^{\frac{1}{n}} < 2; \left\lfloor 100^{\frac{1}{n}} \right\rfloor = 1 \text{ ; } f(n) = 1$$

## DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted with objectivity and without any external influence.
- **Ethical Approval and Consent to Participate:** The content of this article does not necessitate ethical approval or consent to participate with supporting documentation.
- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Authors Contributions:** The authorship of this article is contributed solely.

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2. **"Travaux Dirigés d'Algèbre Générale"** – Dar Elhouda Editions, 2003

My research primarily focuses on number theory, with several publications in renowned international journals:

1. "Existence of Odd Prime Divisors of Sums of Squares of Two Different Non-Zero Integers" – IJR Journal, May 2021- DOI: ijr journal: 10.5281/zenodo.8347261
2. "Factorization of the Sum of Two Distinct Non-Zero Squares into the Product of Two Sums of Two Squares: One of Them is a Prime Number" – IJRAR Journal, October 2023- DOI: ijrar journal:10.1729/journal.43214
3. "Three-Term Arithmetic Progressions Including at Least Two Primes in the Intervals [n², (n+1)²[" – IJSRP Journal, November 2023 – DOI : ijsrp:10.29322/IJSRP.13.11.2033.p14311

With a lifelong dedication to mathematics and education, I continue to explore and contribute to the field of number theory through academic research and scholarly publications.