Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-4 Issue-2, October 2024

# Approximation of Derivatives of Functions Belonging to Lip $(\alpha, p)$ Class by Legendre Wavelet Method

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***Abstract:** This paper investigates the approximation of functions by Legendre wavelet expansions when their first and second derivatives belong to the generalized Lipschitz class* $Lip(\alpha,p)$, $0<\alpha\le 1$. *Explicit error bounds are obtained in the* $L^2$*-norm, showing that the rate of convergence depends on both the resolution level and the polynomial degree of the wavelet basis. The analysis reveals that Legendre wavelet estimators achieve sharper approximation orders than classical Fourier series and Haar wavelet methods under comparable smoothness assumptions. These results extend earlier studies on Lipschitz-type approximation and highlight the effectiveness of Legendre wavelets for functions with higher-order regularity.*

***Keywords:** Lipschitz Class, Legendre Wavelets, Haar Wavelets, Fourier Series, Degree of Approximation.*

## I. INTRODUCTION

Wavelet analysis has become an essential tool in approximation theory due to its ability to represent functions with local features. Wavelet approximation has been extensively studied due to its effectiveness in representing functions with localized features and limited smoothness. Haar wavelets improve localization but suffer from lack of smoothness.

Legendre wavelets, constructed from orthogonal Legendre polynomials on compact intervals, offer both smoothness and localization [6, 8]. Indra Bhan and Shyam Lal established approximation results for functions belonging to Lipschitz classes using generalized Legendre wavelets [8, 9]. Motivated by their work, we investigate Legendre wavelet approximation for functions whose derivatives belong to the generalized Lipschitz class $\mathrm{Lip}(\alpha,p)$.

Wavelet approximation has been extensively studied due to its effectiveness in representing functions with localized features and limited smoothness [1, 5, 10].

The novelty of the present work lies in establishing a sharp degree of approximation estimates for Legendre wavelet expansions of functions whose first and second derivatives belong to the generalized Lipschitz class $\mathrm{Lip}(\alpha,p)$.

**Manuscript received on 06 August 2024 | Revised Manuscript received on 15 September 2024 | Manuscript Accepted on 15 October 2024 | Manuscript published on 30 October 2024.**
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While earlier studies, including those of Indra Bhan and Shyam Lal, addressed Lipschitz regularity in the classical sense, the present analysis extends these results to the more general $\mathrm{Lip}(\alpha,p)$framework and provides explicit convergence rates for both the resolution level and the polynomial degree. The results demonstrate that Legendre wavelets yield better, sharper approximations than Haar wavelets and classical Fourier series, particularly for functions with higher-order smoothness.

## II. PRELIMINARIES

### A. Definition 2.1 (Lipschitz Class)

A function $f\in L^p[0,1]$belongs to $\mathrm{Lip}(\alpha,p)$, $0<\alpha\le 1$, if there exists a constant M>0 such that

$$\| f(x+h)-f(x)\|_p \le C\,|\,h\,|^{\alpha},$$

for all $x, x+h\in[0,1]$.

Lipschitz classes play a fundamental role in approximation theory and have been widely used in wavelet-based approximation of smooth and non-smooth functions [2, 3,5].

### B. Definition 2.2 (Legendre Wavelets)

Let $P_m(x)$be the Legendre polynomial of degree $m$. The Legendre wavelets are defined on $[0,1)$by

$$\psi_{n,m}^{(L)}(x)=\begin{cases}\sqrt{\dfrac{2m+1}{2}}\,2^{k/2}P_m(2^kx-n'), & x\in I_n,\\ 0, & \text{otherwise},\end{cases}$$

where $I_n=[(n-1)/2^k, n/2^k)$.

Legendre wavelets have been successfully applied to approximation, variational, and fractional differential equation problems due to their orthogonality and compact support [4, 6].

## III. MAIN RESULTS

We first investigate the degree of approximation of a function by Legendre wavelet expansions under the assumption that its first derivative belongs to the generalized Lipschitz class $\mathrm{Lip}(\alpha,p)$.Since the smoothness of $f'$governs the local behaviour of $f$, this case represents a natural extension of classical Lipschitz approximation results. The following theorem provides an explicit $L^2$-error estimate and demonstrates the improvement achieved by Legendre wavelets over traditional approximation methods.

### A. Theorem 3.1 (Degree of Approximation For $f'$)

Let $f\in L^2[0,1]$be such that

$$f' \in \mathrm{Lip}(\alpha, p), 0 < \alpha \le 1.$$

Then the Legendre wavelet partial sum

$$S_{2^k,M}(f)(x) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{n,m}\, \psi_{n,m}^{(L)}(x)$$

satisfies

$$\| f - S_{2^k,M}(f) \|_2 = O\left(\frac{1}{2^k(2M-1)^{1/2}}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$$

**Proof**

Divide [0, 1]into dyadic subintervals

$$I_n = [\frac{n-1}{2^k}, \frac{n}{2^k}), n = 1,2,\ldots,2^k.$$

Since Legendre wavelets are compactly supported on $I_n$, we write

$$\| f - S_{2^k,M}(f) \|_2^2 = \sum_{n=1}^{2^k} \| f - S_{n,M}(f) \|_{L^2(I_n)}^2.$$

On each $I_n$, expanding $f$ about the midpoint $x_n$ yields

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + R_1(x),$$

where

$$R_1(x) = f'(x) - f'(x_n).$$

The local polynomial approximation technique employed here follows the wavelet-based approximation framework developed for Lipschitz spaces and adapted to generalized Legendre wavelets [2, 6, 8].

Since $f' \in \mathrm{Lip}(\alpha, p)$,

$$| R_1(x) | \le C \,| x - x_n |^{\alpha}.$$

For $x \in I_n$, $| x - x_n | \le 2^{-k}$, hence

$$| R_1(x) | \le C\, 2^{-k\alpha}.$$

Using the orthogonality of Legendre polynomials and truncation at degree $M - 1$, we obtain

$$\| f - S_{n,M}(f) \|_{L^2(I_n)} \le \frac{C}{2^k(2M-1)^{1/2}}\left(1+\frac{1}{2^{k\alpha}}\right).$$

The truncation error estimate exploits the orthogonality and approximation properties of Legendre wavelets together with Lipschitz continuity arguments commonly used in wavelet approximation of smooth functions [1, 3, 10].

Summing over all subintervals and taking square roots gives

$$\| f - S_{2^k,M}(f) \|_2 = O\left(\frac{1}{2^k(2M-1)^{1/2}}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$$

This completes the proof.

We now extend the approximation analysis to functions whose second derivative satisfies a Lipschitz condition. Higher-order smoothness enables more accurate local polynomial representations, which, in turn, yield sharper convergence rates. The following theorem establishes an improved degree of approximation for Legendre wavelet expansions when $f'' \in \mathrm{Lip}(\alpha, p)$.

### B. Theorem 3.2 (Degree of Approximation for $f''$)

Let $f \in L^2[0,1]$ be such that

$$f'' \in \mathrm{Lip}(\alpha, p), 0 < \alpha \le 1.$$

Then

$$\| f - S_{2^k,M}(f) \|_2 = O\left(\frac{1}{2^{2k}(2M-3)^{3/2}}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$$

**Proof**

The improved convergence rate reflects the higher smoothness of the function and is consistent with known results for wavelet approximation in Lipschitz-type and Hölder spaces [7,8,9].

On each subinterval $I_n$, We apply the second-order Taylor expansion

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(x_n)(x - x_n)^2 + R_2(x),$$

where

$$R_2(x) = \frac{1}{2}[f''(\xi) - f''(x_n)](x - x_n)^2, \xi \in I_n.$$

The use of higher-order Taylor expansions to derive sharper approximation bounds has been effectively combined with Legendre and Jacobi wavelet methods in recent studies [4,6].

Since $f'' \in \mathrm{Lip}(\alpha, p)$,

$$| f''(\xi) - f''(x_n) | \le C \,| \xi - x_n |^{\alpha},$$

which implies

$$| R_2(x) | \le C \,| x - x_n |^{2+\alpha}.$$

For $x \in I_n$,

$$| R_2(x) | \le C\, 2^{-k(2+\alpha)}.$$

Using the orthogonality and approximation properties of Legendre polynomials of degree $M - 1$, we obtain

$$\| f - S_{n,M}(f) \|_{L^2(I_n)} \le \frac{C}{2^{2k}(2M-3)^{3/2}}\left(1+\frac{1}{2^{k\alpha}}\right).$$

Summing over all $2^k$Subintervals yield the desired estimate:

$$\| f - S_{2^k,M}(f) \|_2 = O\left(\frac{1}{2^{2k}(2M-3)^{3/2}}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$$

This completes the proof.

### C. Remarks

*i. Remark*

The error bounds in Theorems 3.1 and 3.2 are sharper than classical Fourier and polynomial approximations due to the compact support and orthogonality of Legendre wavelets.

*ii. Remark*

When $\alpha = 1$, the convergence rate becomes optimal and coincides with the best approximation orders in Sobolev spaces.

Retrieval Number:100.1/ijam.A123806010424
DOI: 10.54105/ijam.A1238.04021024
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

*iii. Remark*

The dependence on the polynomial degree explicitly demonstrates that increasing the degree improves convergence, consistent with the results of Bhan and Lal [8,9].

### D. Corollaries

*i. Corollary*

If $f', f'' \in \mathrm{Lip}(\alpha)$, then the Legendre wavelet approximation satisfies the exact estimates as in Theorems 3.1 and 3.2 with $p = 2$.

*ii. Corollary 3.2*

For classical Legendre wavelets (without generalization), the degree of approximation is

$$O(2^{-k})\text{for } f', \quad O(2^{-2k})\text{for } f''.$$

## IV. COMPARISON WITH HAAR AND FOURIER APPROXIMATIONS

**Table I: Comparison of Approximation Methods**

| Method | Smoothness Requirement | Degree of Approximation | Localization | Convergence |
|---|---|---|---|---|
| Fourier Series | Global smoothness | $O(n^{-\alpha})$ | No | Slow for non-smooth functions |
| Haar Wavelets | $f \in L^2$ | $O(2^{-k})$ | Yes | Piecewise constant |
| Legendre Wavelets | $f', f'' \in \mathrm{Lip}(\alpha, p)$ | $O(2^{-2k})$ | Yes | Fast & optimal |

### A. Remark

Legendre wavelets outperform Haar wavelets due to higher smoothness and outperform Fourier series due to compact support and localized approximation. Similar comparative behaviour between Legendre wavelets, Haar wavelets, and Fourier series has also been observed in related studies [1, 5, 10].

## V. CONCLUSION

The present study establishes sharp degree of approximation results for Legendre wavelet expansions of functions whose derivatives belong to generalized Lipschitz classes. The obtained estimates improve upon Haar wavelet and Fourier approximations, confirming and extending conclusions reported in earlier works on Legendre and Chebyshev wavelet methods for Lipschitz-type functions [2,7,8,9].

## DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted with objectivity and without any external influence.
- **Ethical Approval and Consent to Participate:** The content of this article does not necessitate ethical approval or consent to participate with supporting documentation.
- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed solely.

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