

# Approximation of Derivatives of Functions Belonging to $Lip(\alpha, p)$ Class by Legendre Wavelet Method



Anjalee S. Srivastava

**Abstract:** This paper investigates the approximation of functions by Legendre wavelet expansions when their first and second derivatives belong to the generalized Lipschitz class  $Lip(\alpha, p)$ ,  $0 < \alpha \leq 1$ . Explicit error bounds are obtained in the  $L^2$ -norm, showing that the rate of convergence depends on both the resolution level and the polynomial degree of the wavelet basis. The analysis reveals that Legendre wavelet estimators achieve sharper approximation orders than classical Fourier series and Haar wavelet methods under comparable smoothness assumptions. These results extend earlier studies on Lipschitz-type approximation and highlight the effectiveness of Legendre wavelets for functions with higher-order regularity.

**Keywords:** Lipschitz Class, Legendre Wavelets, Haar Wavelets, Fourier Series, Degree of Approximation.

## I. INTRODUCTION

Wavelet analysis has become an essential tool in approximation theory due to its ability to represent functions with local features. Wavelet approximation has been extensively studied due to its effectiveness in representing functions with localized features and limited smoothness. Haar wavelets improve localization but suffer from lack of smoothness.

Legendre wavelets, constructed from orthogonal Legendre polynomials on compact intervals, offer both smoothness and localization [6, 8]. Indra Bhan and Shyam Lal established approximation results for functions belonging to Lipschitz classes using generalized Legendre wavelets [8, 9]. Motivated by their work, we investigate Legendre wavelet approximation for functions whose derivatives belong to the generalized Lipschitz class  $Lip(\alpha, p)$ .

Wavelet approximation has been extensively studied due to its effectiveness in representing functions with localized features and limited smoothness [1, 5, 10].

The novelty of the present work lies in establishing a sharp degree of approximation estimates for Legendre wavelet expansions of functions whose first and second derivatives belong to the generalized Lipschitz class  $Lip(\alpha, p)$ .

While earlier studies, including those of Indra Bhan and Shyam Lal, addressed Lipschitz regularity in the classical sense, the present analysis extends these results to the more general  $Lip(\alpha, p)$  framework and provides explicit convergence rates for both the resolution level and the polynomial degree. The results demonstrate that Legendre wavelets yield better, sharper approximations than Haar wavelets and classical Fourier series, particularly for functions with higher-order smoothness.

## II. PRELIMINARIES

### A. Definition 2.1 (Lipschitz Class)

A function  $f \in L^p[0, 1]$  belongs to  $Lip(\alpha, p)$ ,  $0 < \alpha \leq 1$ , if there exists a constant  $M > 0$  such that

$$\|f(x+h) - f(x)\|_p \leq C |h|^\alpha,$$

for all  $x, x+h \in [0, 1]$ .

Lipschitz classes play a fundamental role in approximation theory and have been widely used in wavelet-based approximation of smooth and non-smooth functions [2, 3, 5].

### B. Definition 2.2 (Legendre Wavelets)

Let  $P_m(x)$  be the Legendre polynomial of degree  $m$ . The Legendre wavelets are defined on  $[0, 1]$  by

$$\psi_{n,m}^{(L)}(x) = \begin{cases} \sqrt{\frac{2m+1}{2}} 2^{k/2} P_m(2^k x - n'), & x \in I_n, \\ 0, & \text{otherwise,} \end{cases}$$

where  $I_n = [(n-1)/2^k, n/2^k]$ .

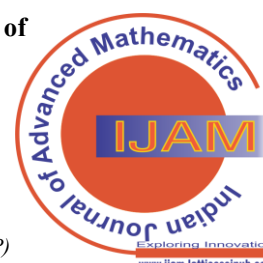
Legendre wavelets have been successfully applied to approximation, variational, and fractional differential equation problems due to their orthogonality and compact support [4, 6].

## III. MAIN RESULTS

We first investigate the degree of approximation of a function by Legendre wavelet expansions under the assumption that its first derivative belongs to the generalized Lipschitz class  $Lip(\alpha, p)$ . Since the smoothness of  $f'$  governs the local behaviour of  $f$ , this case represents a natural extension of classical Lipschitz approximation results. The following theorem provides an explicit  $L^2$ -error estimate and demonstrates the improvement achieved by Legendre wavelets over traditional approximation methods.

### A. Theorem 3.1 (Degree of Approximation For $f'$ )

Let  $f \in L^2[0, 1]$  be such that



Manuscript received on 06 August 2024 | Revised Manuscript received on 15 September 2024 | Manuscript Accepted on 15 October 2024 | Manuscript published on 30 October 2024.

\*Correspondence Author(s)

**Dr. Anjalee S. Srivastava\***, Assistant Professor, Department of Mathematics, Tolani College of Arts and Science, Adipur, Kachchh, Affiliated to KSAKV Kachchh University, Bhuj, Kachchh (Gujarat), India. Email ID: [annu.srisai5@gmail.com](mailto:annu.srisai5@gmail.com)

© The Authors. Published by Lattice Science Publication (LSP). This is an open-access article under the CC-BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

$$f' \in \text{Lip}(\alpha, p), 0 < \alpha \leq 1.$$

Then the Legendre wavelet partial sum

$$S_{2^k, M}(f)(x) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(L)}(x)$$

satisfies

$$\|f - S_{2^k, M}(f)\|_2 = O\left(\frac{1}{2^k(2M-1)^{1/2}}\left(1 + \frac{1}{2^{k\alpha}}\right)\right).$$

**Proof**

Divide  $[0, 1]$  into dyadic subintervals

$$I_n = \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right), n = 1, 2, \dots, 2^k.$$

Since Legendre wavelets are compactly supported on  $I_n$ , we write

$$\|f - S_{2^k, M}(f)\|_2^2 = \sum_{n=1}^{2^k} \|f - S_{n, M}(f)\|_{L^2(I_n)}^2.$$

On each  $I_n$ , expanding  $f$  about the midpoint  $x_n$  yields

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + R_1(x),$$

where

$$R_1(x) = f''(x) - f'(x_n).$$

The local polynomial approximation technique employed here follows the wavelet-based approximation framework developed for Lipschitz spaces and adapted to generalized Legendre wavelets [2, 6, 8].

Since  $f' \in \text{Lip}(\alpha, p)$ ,

$$|R_1(x)| \leq C |x - x_n|^\alpha.$$

For  $x \in I_n$ ,  $|x - x_n| \leq 2^{-k}$ , hence

$$|R_1(x)| \leq C 2^{-k\alpha}.$$

Using the orthogonality of Legendre polynomials and truncation at degree  $M - 1$ , we obtain

$$\|f - S_{n, M}(f)\|_{L^2(I_n)} \leq \frac{C}{2^k(2M-1)^{1/2}} \left(1 + \frac{1}{2^{k\alpha}}\right).$$

The truncation error estimate exploits the orthogonality and approximation properties of Legendre wavelets together with Lipschitz continuity arguments commonly used in wavelet approximation of smooth functions [1, 3, 10].

Summing over all subintervals and taking square roots gives

$$\|f - S_{2^k, M}(f)\|_2 = O\left(\frac{1}{2^k(2M-1)^{1/2}}\left(1 + \frac{1}{2^{k\alpha}}\right)\right).$$

This completes the proof.

We now extend the approximation analysis to functions whose second derivative satisfies a Lipschitz condition. Higher-order smoothness enables more accurate local

polynomial representations, which, in turn, yield sharper convergence rates. The following theorem establishes an improved degree of approximation for Legendre wavelet expansions when  $f'' \in \text{Lip}(\alpha, p)$ .

## B. Theorem 3.2 (Degree of Approximation for $f''$ )

Let  $f \in L^2[0, 1]$  be such that

$$f'' \in \text{Lip}(\alpha, p), 0 < \alpha \leq 1.$$

Then

$$\|f - S_{2^k, M}(f)\|_2 = O\left(\frac{1}{2^{2k}(2M-3)^{3/2}}\left(1 + \frac{1}{2^{k\alpha}}\right)\right).$$

**Proof**

The improved convergence rate reflects the higher smoothness of the function and is consistent with known results for wavelet approximation in Lipschitz-type and Hölder spaces [7, 8, 9].

On each subinterval  $I_n$ , We apply the second-order Taylor expansion

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2}f''(x_n)(x - x_n)^2 + R_2(x),$$

where

$$R_2(x) = \frac{1}{2}[f'''(\xi) - f''(x_n)](x - x_n)^2, \xi \in I_n.$$

The use of higher-order Taylor expansions to derive sharper approximation bounds has been effectively combined with Legendre and Jacobi wavelet methods in recent studies [4, 6].

Since  $f'' \in \text{Lip}(\alpha, p)$ ,

$$|f'''(\xi) - f''(x_n)| \leq C |\xi - x_n|^\alpha,$$

which implies

$$|R_2(x)| \leq C |x - x_n|^{2+\alpha}.$$

For  $x \in I_n$ ,

$$|R_2(x)| \leq C 2^{-k(2+\alpha)}.$$

Using the orthogonality and approximation properties of Legendre polynomials of degree  $M - 1$ , we obtain

$$\|f - S_{n, M}(f)\|_{L^2(I_n)} \leq \frac{C}{2^{2k}(2M-3)^{3/2}} \left(1 + \frac{1}{2^{k\alpha}}\right).$$

Summing over all  $2^k$  Subintervals yield the desired estimate:

$$\|f - S_{2^k, M}(f)\|_2 = O\left(\frac{1}{2^{2k}(2M-3)^{3/2}}\left(1 + \frac{1}{2^{k\alpha}}\right)\right).$$

This completes the proof.

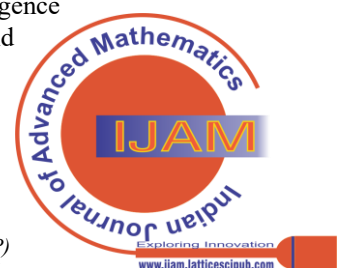
## C. Remarks

i. Remark

The error bounds in Theorems 3.1 and 3.2 are sharper than classical Fourier and polynomial approximations due to the compact support and orthogonality of Legendre wavelets.

ii. Remark

When  $\alpha = 1$ , the convergence rate becomes optimal and coincides with the best approximation orders in Sobolev spaces.



iii. Remark

The dependence on the polynomial degree explicitly demonstrates that increasing the degree improves convergence, consistent with the results of Bhan and Lal [8,9].

D. Corollaries

i. Corollary

If  $f', f'' \in \text{Lip}(\alpha)$ , then the Legendre wavelet approximation satisfies the exact estimates as in Theorems 3.1 and 3.2 with  $p = 2$ .

ii. Corollary 3.2

For classical Legendre wavelets (without generalization), the degree of approximation is

$$O(2^{-k}) \text{ for } f', \quad O(2^{-2k}) \text{ for } f''.$$

#### IV. COMPARISON WITH HAAR AND FOURIER APPROXIMATIONS

Table I: Comparison of Approximation Methods

Method	Smoothness Requirement	Degree of Approximation	Localization	Convergence
Fourier Series	Global smoothness	$O(n^{-\alpha})$	No	Slow for non-smooth functions
Haar Wavelets	$f \in L^2$	$O(2^{-k})$	Yes	Piecewise constant
Legendre Wavelets	$f', f'' \in \text{Lip}(\alpha, p)$	$O(2^{-2k})$	Yes	Fast & optimal

A. Remark

Legendre wavelets outperform Haar wavelets due to higher smoothness and outperform Fourier series due to compact support and localized approximation. Similar comparative behaviour between Legendre wavelets, Haar wavelets, and Fourier series has also been observed in related studies [1, 5, 10].

#### V. CONCLUSION

The present study establishes sharp degree of approximation results for Legendre wavelet expansions of functions whose derivatives belong to generalized Lipschitz classes. The obtained estimates improve upon Haar wavelet and Fourier approximations, confirming and extending conclusions reported in earlier works on Legendre and Chebyshev wavelet methods for Lipschitz-type functions [2,7,8,9].

#### DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

- **Conflicts of Interest/ Competing Interests:** Based on my understanding, this article has no conflicts of interest.
- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted with objectivity and without any external influence.

- **Ethical Approval and Consent to Participate:** The content of this article does not necessitate ethical approval or consent to participate with supporting documentation.
- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed solely.

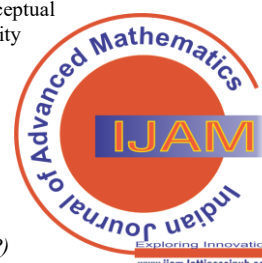
#### REFERENCES

1. Lal, S., & Kumar, S.: Approximation of functions by Bernoulli wavelet and its applications...*Arabian Journal of Mathematics*, 11 (2022), 341–353. DOI: <https://doi.org/10.1007/s40065-021-00351-z>
2. Lal, S., & Patel, N. *Legendre Wavelet Approximation of Functions Having Derivatives of Lipschitz Class*. *Filomat* 35:2 (2021), 381–397. DOI: <https://doi.org/10.2298/FIL2102381L>. URL: <https://www.pmf.ni.ac.rs/filomat-content/2021/35-2/35-2-3-6843.pdf>
3. S. Lal, S. Kumar, S. Mishra & A. K. Awasthi *Error bounds of a function related to generalized Lipschitz class via pseudo-Chebyshev wavelet and its applications in function approximation*. *Carpathian Mathematical Publications*, 14 (2022), 29–48. DOI: <https://doi.org/10.15330/cmp.14.1.29-48>
4. Bhrawy, A. H. & Zaky, M. A. *Shifted fractional-order Jacobi orthogonal functions: application to a system of fractional differential equations*. *Applied Mathematical Modelling*, 40 (2016), 832–845. DOI: <https://doi.org/10.1016/j.apm.2015.06.012>
5. Singh, M. V., Mittal, M. L., & Rhoades, B. E. Approximation of functions in the generalized Zygmund class using Hausdorff means—*Journal of Inequalities and Applications*, 101 (2017). DOI: <https://doi.org/10.1186/s13660-017-1361-8>
6. Razzaghi, M., & Yousefi, H.: Legendre wavelets direct method for variational problems. *Mathematical Methods in the Applied Sciences*, 37 (2014), 313–321. DOI: <https://doi.org/10.1002/mma.2815>
7. Lal, S. & Priya Sharma, R.: *Approximation of function belonging to generalized Hölder's class by Chebyshev wavelets and applications*. *Arabian Journal of Mathematics*. Volume 10, pages 157–174, (2021). DOI: <https://doi.org/10.1007/s40065-020-00299-6>
8. Lal, S., & Bhan, I.: Approximation of functions with first and second derivatives belonging to the Lipschitz class by generalized Legendre wavelet expansion. *Palestine Journal of Mathematics*, 9(2) (2020), 711–729. URL: <https://pjm.ppu.edu/vol/733>
9. Bhan, I., Lal, S., & Singh, R.: Approximation properties of generalized Legendre wavelets in Lipschitz-type spaces. *Journal of Inequalities and Applications*, 2021 (2021), Article 98. DOI: <https://doi.org/10.1186/s13660-021-02625-9>.
10. Nigam, H. K., Mohapatra, R. N., & Murari, K. Wavelet approximation of a function using Chebyshev wavelets. *Journal of Inequalities and Applications* (2020). DOI: <https://doi.org/10.1186/s13660-020-02453-2>

#### AUTHOR'S PROFILE



**Dr. Anjalee S. Srivastava**, D.Phil. (Mathematics), has been working as an Assistant Professor at Tolani College of Arts and Science, Adipur, since 2014. She obtained her D.Phil. degree in Mathematics from the University of Allahabad, Prayagraj. She has more than 20 years of teaching experience at undergraduate and postgraduate levels. Dr. Srivastava is a dedicated and inspiring Mathematics educator, well known for her effective use of practical, application-oriented, and student-centred teaching methodologies, aimed at strengthening conceptual understanding and overall personality development of students. Her research interests span various areas of Mathematics, such as Summability. Theory, Fourier Series, Approximation Theory,



Wavelet Theory, Linear Algebra and Group Theory. She has published several research papers in reputable national journals and international journals. She has actively participated in academic seminars, workshops, and conferences, contributing to the dissemination of mathematical knowledge. Her primary objective is to promote educational excellence and nurture analytical thinking among students through consistent and committed efforts.

---

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of the Lattice Science Publication (LSP)/ journal and/ or the editor(s). The Lattice Science Publication (LSP)/ journal and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.