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On the Polynomial Structure of $r_k(n)$

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Abstract: ***If $r_k(n)$ is the number of representations of a positive integer n as the sum of k squares, then $r_k(n)$ is a polynomial in k of degree n; here we exhibit expressions for certain coefficients through this polynomial.***

Keywords: Sum of Divisors Function, Sequences of Integers, Sums of Squares, Recurrence Relations.

I. INTRODUCTION

In [1][17][18][19][20][21] is proved the following recurrence relation for $r_k(n)$,the number of representations of n as a sum of k squares [2-4]:

$$n\, r_k(n) = 2k \sum_{j=1}^{n} A(j)\, r_k(n-j), \qquad (1)$$

such that:

$$A(j) = (-1)^{j-1}\, j \sum_{odd\ d/j} \frac{1}{d}, \qquad (2)$$

then is easy to obtain the values $A(1) = 1,\ A(2) = -2,\ A(3) = 4,\ A(4) = -4,\ etc.$, thus it appears the sequence studied in [5]:

1, -2, 4, -4, 6, -8, 8, -8, 13, -12, 12, -16, 14, -16, 24, -16, 18, -26, 20, -24, 32, -24, 24, -32, 31, -28……., (3)

associated to $A186690\ (n) = (-1)^{n-1}\ A002131\ (n)$ [6, 7], hence:

$$A(n) = \begin{cases} -\left(\sigma(n) - \sigma\left(\frac{n}{2}\right)\right), & n \text{ is even}, \\ \sigma(n), & n \text{ is odd}, \end{cases} \qquad (4)$$

involving the sum of divisors function $\sigma(n)$ [8-10].

From (1) it is clear that $r_k(n)$ is a polynomial in k of degree n:

$$r_k(n) = a(n,n)\, k^n + a(n,n-1)k^{n-1} + \cdots + a(n,2)k^2 + a(n,1)\, k\,, \qquad (5)$$

Thus it is interesting to deduce explicit expressions for the coefficients of this polynomial.

II. EXPRESSIONS FOR $a(n, n-m)$

The use of (5) in (1) generates recurrence relations between the coefficients of (5), which can be solved applying the Zeta-transform to obtain the useful relations:

$$a(n,n) = \frac{2^n}{n!}, \quad n \geq 1, \qquad a(n,1) = \frac{2}{n}\, A(n), \quad n \geq 1,$$

$$a(n,n-1) = -\frac{2^{n-1}}{(n-2)!}, \quad n \geq 2, \qquad a(n,n-2) = \frac{2^{n-3}\,(3\,n-1)}{3\,(n-3)!}, \quad n \geq 3, \qquad (6)$$

$$a(n,n-3) = \frac{2^{n-4}(n+2)\,(3-n)}{3\,(n-4)!}, \quad n \geq 4, \quad a(n,n-4) = \frac{2^{n-7}}{45}\left[\frac{8(85\,n-371)}{(n-5)!} + \frac{15(n+9)}{(n-7)!}\right], \quad n \geq 5, etc.$$

which allow to reproduce several polynomials type (5) reported in the literature, for example [8, 11]:

$$r_k(1) = 2k, \qquad r_k(2) = 2k(k-1), \qquad r_k(3) = \frac{4}{3}\, k(k-1)(k-2),$$

$$r_k(4) = \frac{2}{3}\, k[3(2k-1) + k(k-1)(k-5)], \quad r_k(5) = \frac{4}{15}\, k(k-1)[3(2k-3) + k(k-4)(k-5)], \qquad (7)$$

$$r_k(6) = \frac{4}{45}\, k(k-1)(k-2)[45 + (k-3)(k-4)(k-5)],$$

$$r_k(7) = \frac{8}{315}\, k(k-1)(k-2)(k-3)(k^3 - 15k^2 + 74k - 15), \qquad \ldots$$

In [12] it was shown that the $a(n,m)$ can be written in terms of partial exponential Bell polynomials [13-15].

Remark: The solution of (1) is given by [16]:

$$r_k(n) = \frac{1}{n!} B_n\left(2k \cdot\ 0!\, A(1), 2k \cdot 1!\, A(2), 2k \cdot 2!\, A(3), \ldots, 2k \cdot (n-1)!\, A(n)\right), \qquad (8)$$

involving the complete Bell polynomials.

III. CONCLUSION

The primary objective of this paper is to express the structure of number of representations of a positive integer n as the sum of k squares namely $r_k(n)$ in terms of polynomials whose coefficients forms interesting class of numbers. In particular we have expressed in terms of a polynomial whose coefficients depends on sum of divisors function $\sigma(n)$ as presented in (5).

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Applying zeta transforms for $a(n, n-m)$, we have obtained nice polynomial expressions for $r_k(n)$ from $n = 1$ to 7 both inclusive. From these expressions, we notice from (7), that $r_k(n)$ is a polynomial of degree n in k. Finally through expression (8), we have presented the connection $r_k(n)$ and the complete Bell polynomials. These novel expressions may provide additional insights to already existing vast literature regarding the structure of $r_k(n)$.

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Authors Contributions	All authors have equal participation in this article.

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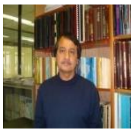

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