Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-3 Issue-1, April 2023

Derivation and Implementation of a Fifth Stage Fourth Order Explicit Runge-Kutta Formula using $f(x,y)$ Functional Derivatives

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***Abstract:** This paper is aimed at using $f(x,y)$ functional derivatives to derive a fifth stage fourth order Explicit Runge-Kutta formula for solving initial value problems in Ordinary Differential Equations. The $f(x,y)$ functional derivatives from the general Runge-Kutta scheme will be compared with the $f(x,y)$ functional derivatives from the Taylor series expansion to derive the method. The method will be implemented on some initial value problems, and results compared with results from the classical fourth order method. The results revealed that the method compared favorably well with the existing classical fourth order method.*

***Key words:** initial value problems, Comparison, explicit, $f(x,y)$ partial derivatives, Explicit Runge-Kutta Methods, Linear and non- linear equations, Taylor series expansion.*

I. INTRODUCTION

This paper centers on separating the $f(x,y)$ functional derivatives from the $f(y)$ functional derivatives, after using the Taylor series expansion. This is aimed at generating a fifth stage fourth order Explicit method that will improve results. Scientific implementation of the formula on initial-value problems of the form: $y^1 = f(x,y),\ \ y(x_0) = y_0,\ a \le x \le b$, is also considered with a view of studying its performance. The essence is to see if the formula can improve results. Recent works on Runge-Kutta analysis are seen in [1], [2][14] and [9]. More recent works are that of [8], [10] and [11]. The work of [3], [4] and [5] revealed much successes in the analysis of explicit Runge-Kutta methods and their various modifications and transformations. Conclusively, despite the fact that good, reliable explicit Runge-Kutta formulae exist, there is still need for their transformation to rooted tree diagrams as seen in the works of [6] [15][16] [12] [13]and [7].

II. METHOD OF DERIVATION

i. From the general Runge-Kutta method, get a Fifth Stage-Fourth order method,
ii. Obtain the Taylor series expansion of $k_{i's}^{i}$ about the point (x_n, y_n), i=2,3,4,5
iii. Carry out substitution to ensure that all the $k_{i's}^{i}$ are in terms of k_1 only.
iv. Insert the $k_{i's}^{i}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5$
v. Separate all $f(x,y)$ functional derivatives with their coefficients from all $f(y)$ functional derivatives with their coefficients.
vi. Compare the coefficients of all $f(x,y)$ functional derivatives with the Taylor series expansion involving only $f(x,y)$ functional derivatives with their coefficient of the form:

$$\phi(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}(f_{xx} + 2ff_{xy} + f_xf_y) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

vii. As a result, a set of linear/non-linear equations will be generated. Resolve the set of equations to get a fifth stage fifth order Explicit Runge Kutta formula.

III. DERIVATION OF THE FIFTH STAGE FOURTH–ORDER ERK METHOD

From the scheme in (3.4.1), the explicit fifth-stage fourth-order method is given below:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5)$$
$$k_1 = f\,(x_n, y_n)$$
$$k_2 = f\,(x_n + c_2, y_n + ha_{21}k_1)$$
$$k_3 = f\,(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f\,(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3))$$
$$k_5 = f\,(x_n + c_5h, y_n + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)) \quad (3.1)$$

Using Taylor's series expansion for $k_i's$, we have:

$$k_1 = f\,(x_n, y_n)$$
$$k_2 = \sum_{r=0}^{\infty}\frac{1}{r!}\left(c_2h\frac{\partial}{\partial x} + ha_{21}k_1\frac{\partial}{\partial y}\right)^r f(x_n, y_n)$$
$$k_3 = \sum_{r=0}^{\infty}\frac{1}{r!}\left(c_3h\frac{\partial}{\partial x} + h(a_{31}k_1 + a_{32}k_2)\frac{\partial}{\partial y}\right)^r f(x_n, y_n)$$
$$k_4 = \sum_{r=0}^{\infty}\frac{1}{r!}\left(c_4h\frac{\partial}{\partial x} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{\partial}{\partial y}\right)^r f(x_n, y_n)$$

Manuscript received on 16 February 2023 | Revised Manuscript received on 21 March 2023 | Manuscript Accepted on 15 April 2023 | Manuscript published on 30 December 2023.
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$k_5 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_5 h\frac{\partial}{\partial x} + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)\frac{\partial}{\partial y})^r f(x_n, y_n)$ (3.2)

Hence, we have:

$k_1 = f$

$k_2 = f + (c_2hf_x + ha_{21}k_1f_y) + \frac{1}{2!}(c_2hf_x + ha_{21}k_1f_y)^2 + \frac{1}{3!}(c_2hf_x + ha_{21}k_1f_y)^3 + \frac{1}{4!}(c_2hf_x + ha_{21}k_1f_y)^4 + 0(h^5)$

$k_3 = f + (c_3hf_x + h(a_{31}k_1 + a_{32}k_2) f_y) + \frac{1}{2!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2) f_y)^2 + \frac{1}{3!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2) f_y)^3 + \frac{1}{4!}(c_3hf_x + h(a_{31}k_1 + a_{32}k_2) f_y)^4 + 0(h^5)$

$k_4 = f + (c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3) f_y) + \frac{1}{2!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3) f_y)^2 + \frac{1}{3!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3) f_y)^3 + \frac{1}{4!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3) f_y)^4 + 0(h^5)$

$k_5 = f + (c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4) f_y) + \frac{1}{2!}(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4) f_y)^2 + \frac{1}{3!}(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4) f_y)^3 + \frac{1}{4!}(c_5hf_x + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4) f_y)^4 + 0(h^5)$ (3.3)

Expanding fully and substituting the various k_i' s, i = 2, 3, 4, 5 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives and (x, y) derivatives separately, we have:

$k_1 = f$

$k_2 = f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy} + hc_2f_x + \frac{h^2}{2!}c_2^2f_{xx} + h^2c_2a_{21}ff_{xy}\ \frac{h^3}{3!}c_2^3f_{xxx} + \frac{h^3}{2!}c_2^2a_{21}ff_{xxy} + \frac{h^3}{2!}c_2a_{21}^2f^2f_{xyy} + \frac{h^4}{4!}c_2^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{21}ff_{xxxy}\ \frac{h^4}{2!2!}c_2^2a_{21}^2f^2f_{xxyy} + \frac{h^4}{3!}c_2a_{21}^3f^3f_{xyyy} + 0\,(h^5)$

$k_3 = f + h\,(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + \frac{h^3}{3!}a_{21}a_{32}(a_{21} + 2(a_{31} + a_{32}))f^2f_yf_{yy} + \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3) f^3f_{yyy} + \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^3a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \frac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2 + \frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy} + hc_3f_x + \frac{h^2}{2!}c_3^2f_{xx} + h^2c_3(a_{31} + a_{32})ff_{xy} + h^2c_2a_{32}f_xf_y + \frac{h^3}{3!}c_3^3f_{xxx} + \frac{h^3}{2!}c_3^2(a_{31} + a_{32})ff_{xxy} + \frac{h^3}{2!}c_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xyy} + h^3c_2a_{32}(a_{31} + a_{32})ff_xf_{yy} + h^3a_{21}a_{32}(c_2 + c_3)ff_yf_{xy} + \frac{h^3}{2!}c_2^2a_{32}f_yf_{xx} + h^3c_2c_3a_{32}f_xf_{xy} + \frac{h^4}{4!}c_3^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{32}f_{xxx}f_y + \frac{h^4}{2!}c_3^2c_2a_{32}f_xf_{xxy} + \frac{h^4}{2!}a_{21}a_{32}(c_2^2 + c_3^2)ff_yf_{xxy} + \frac{h^4}{3!}a_{21}a_{32}(2c_2a_{31} + 3c_2a_{21} + 6c_3a_{31})f^2f_yf_{xyy} + \frac{h^4}{2!}c_3a_{32}c_2^2f_{xx}f_{xy} + \frac{h^4}{2!}c_2^2a_{32}(a_{31} + a_{32})ff_{xx}f_{yy} + \frac{h^4}{2!}a_{21}a_{32}(2c_2a_{31} + 2c_2a_{32} + c_3a_{21})f^2f_{xy}f_{yy} + h^4c_3a_{32}c_2a_{21}ff_{xy}^2 + \frac{h^4}{2!}a_{32}^2c_2^2f_x^2f_{yy} + h^4a_{32}^2a_{21}c_2ff_xf_yf_{yy} + \frac{h^4}{2!}c_3c_2a_{32}(6a_{31} + 2a_{32})ff_xf_{xyy} + \frac{h^4}{2!}c_2a_{32}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_xf_{yyy} + \frac{h^4}{3!}c_3^3(a_{31} + a_{32})ff_{xxxy} + \frac{h^4}{2!2!}c_3^2(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xxyy} + \frac{h^4}{3!}c_3(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{xyyy} + 0\,(h^5)$

$k_4 = f + h\,(a_{41} + a_{42} + a_{43})ff_y + h^2(a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43})ff_y^2 + \frac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2)f^2f_{yy} + \frac{h^3}{2!}(a_{21}^2a_{42} + a_{31}^2a_{43} + 2a_{31}a_{32}a_{43} + a_{32}^2a_{43} + 2a_{21}a_{41}a_{42} + 2a_{31}a_{41}a_{43} + 2a_{32}a_{41}a_{43} + 2a_{31}a_{42}a_{43} + 2a_{32}a_{42}a_{43} + 2a_{21}a_{42}a_{43} + 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2)\, f^2f_yf_{yy} + h^3a_{21}a_{32}a_{43}ff_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{31}^3a_{43} + 3a_{31}^2a_{32}a_{43} + 3a_{31}a_{32}^2a_{43} + 3a_{21}a_{41}^2a_{42} + 3a_{31}a_{41}^2a_{43} + 3a_{32}a_{41}^2a_{43} + 6a_{21}a_{41}a_{42}^2 + 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} +$

Retrieval Number:100.1/ijam.A1144043123
DOI: 10.54105/ijam.A1144.043123
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

$3a_{31}a_{41}^2a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2a_{43} + 6a_{21}a_{42}^2a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 +$
$3a_{21}a_{42}a_{43}^2 + 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3) + \frac{h^4}{2!}(a_{21}^2a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2a_{43} + 2a_{21}a_{32}a_{41}a_{43} +$
$2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2a_{42}^2 + a_{21}a_{32}a_{43}^2 + a_{31}^2a_{43}^2 + 2a_{31}a_{32}a_{43}^2 +$
$a_{32}^2a_{43}^2)f^2f_y^2f_{yy} + \frac{h^4}{2!}(a_{21}^2a_{41}a_{42} + a_{31}^2a_{41}a_{42} + 2a_{31}a_{32}a_{41}a_{43} + a_{32}^2a_{41}a_{43} + a_{31}^2a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} +$
$a_{32}^2a_{42}a_{43} + \frac{a_{21}^2a_{42}^2}{2!} + \frac{a_{31}^2a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 + \frac{a_{31}^2a_{43}^2}{2!})f^3f_{yy}^2 + \frac{h^4}{4!}(a_{41}^4 + 4_{41}^3a_{42} + 4a_{41}^3a_{43} + 6a_{41}^2a_{41}^2 +$
$12a_{41}^2a_{42}a_{43} + 6a_{42}^2a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 + 12a_{41}a_{42}^2a_{43} + 2a_{41}a_{42}a_{43}^2 +$
$6a_{41}^2a_{43}^2 + 4a_{42}^3a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4)\, f^4f_{yyyy} + hc_4f_x + h^2(c_4a_{42} + c_3a_{43})f_xf_y + \frac{h^2}{2!}c_4^2f_{xx} + h^2c_4(a_{41} + a_{42} +$
$a_{43})ff_{xy} + \frac{h^3}{2!}(c_2^2a_{42} + c_3^2a_{43})f_{xx}f_y + h^2(c_4a_{21}a_{42} +$
$c_3a_{31}a_{43} + c_3a_{32}a_{43})ff_{xy}f_y + h^3c_2a_{32}a_{43}f_xf_y^2 + h^3(c_2c_4a_{42} + c_3c_4a_{43})\, f_xf_{xy}$
$+h^3c_2a_{32}a_{43}f_xf_y^2 + h^3(c_2c_4a_{42} + c_3c_4a_{43})\, f_xf_{xy} + h^3(c_2a_{21}a_{42} + c_4a_{31}a_{43} +$
$c_4a_{32}a_{43})ff_yf_{xy} + h^3(c_2a_{41}a_{42} + c_3a_{41}a_{43} + c_3a_{42}a_{43} + c_2a_{42}a_{43} + c_2a_{42}^2 +$
$c_3a_{43}^2)ff_xf_{yy} + \frac{h^3}{3!}c_4^3f_{xxx} + \frac{h^3}{2!}(c_4^2a_{41} + c_4^2a_{42} + c_4^2a_{43})ff_{xxy} + \frac{h^3}{2!}c_4(a_{41}^2 + 2a_{41}a_{42} +$
$2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2f_{xyy} + \frac{h^4}{3!}(c_2^3a_{42} +$
$c_3^3a_{43})f_{xxx}f_y + \frac{4}{3!}(3c_2^2a_{21}a_{42} + c_3^2a_{31}a_{43} + 3c_3^2a_{32}a_{43} + 3c_4^2a_{21}a_{42} + 3c_4^2a_{31}a_{43} +$
$3c_4^2a_{32}a_{43})ff_{xxy}f_y + \frac{h^4}{2!}(c_2a_{21}^2a_{42} + 2c_3a_{31}a_{32}a_{43} + c_3a_{31}^2a_{43} + c_3a_{32}^2a_{43} +$
$2c_4a_{21}a_{41}a_{42} + 2c_4a_{31}a_{41}a_{43} + 2c_4a_{32}a_{41}a_{43} + 2c_4a_{21}a_{42}^2 + 2c_4a_{31}a_{42}a_{43} +$
$2c_4a_{32}a_{42}a_{43} + 2c_4a_{21}a_{42}a_{43} + 2c_4a_{31}a_{43}^2 + 2c_4a_{32}a_{43}^2)f^2f_yf_{xyy} +$
$\frac{h^4}{3!}(c_2^2a_{32}a_{43})f_{xx}f_y^2 + h^4(c_2a_{21}a_{32}a_{43} + c_3a_{21}a_{32}a_{43} + c_4a_{21}a_{32}a_{43})ff_{xy}f_y^2 +$
$\frac{h^4}{2!}(2c_2a_{31}a_{32}a_{43} + 2c_2a_{32}^2a_{43} + 2c_2a_{32}a_{41}a_{43} + 2c_2a_{32}a_{42}a_{43} + 2c_2a_{31}a_{42}a_{43} +$
$2c_2a_{32}a_{42}a_{43} + 2c_3a_{21}a_{42}a_{43} + 2c_2a_{21}a_{42}^2 + c_2a_{32}a_{43}^2 + c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2 +$
$c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2)ff_xf_yf_{yy} + h^4(c_2c_3a_{32}a_{43} + c_2c_4a_{32}a_{43})f_xf_yf_{xy} +$
$\frac{h^4}{2!}(c_2^2c_4a_{42} + c_3^2c_4a_{43})f_{xx}f_{xy} + h^4(c_2c_4a_{21}a_{42} + c_3c_4a_{31}a_{43} + c_3c_4a_{32}a_{43})$
$ff_{xy}^2 + \frac{h^4}{2!}(c_4a_{21}^2a_{42} + c_4a_{31}^2a_{43} + 2c_4a_{31}a_{32}a_{43} + c_4a_{32}^2a_{43} + 2c_2a_{21}a_{41}a_{42} +$
$2c_3a_{31}a_{41}a_{43} + 2c_3a_{32}a_{41}a_{43} + 2c_3a_{31}a_{42}a_{43} + 2c_3a_{32}a_{42}a_{43} + c_2a_{21}a_{42}^2 + c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2)\, f^2f_{xy}f_{yy} +$
$\frac{h^4}{2!}(2c_2c_3a_{42}a_{43} + c_2^2a_{42}^2 + c_3^2a_{43}^2)f_x^2f_{yy} + \frac{h^4}{2!}(c_2c_4^2a_{42} +$
$c_3c_4^2a_{43})f_xf_{xxy} + h^4(c_2c_4a_{41}a_{42} + c_3c_4a_{41}a_{43} + c_2c_4a_{42}^2 + c_3c_4a_{42}a_{43}) +$
$c_2c_4a_{42}a_{43} + c_3c_4a_{43}^2)ff_xf_{xyy} + \frac{h^4}{2!}(c_2c_{41}^2a_{42} + c_3c_{41}^2a_{43} + 2c_2a_{41}a_{42}^2 + 2c_3a_{41}a_{42}a_{43} + 2c_2a_{41}a_{42}a_{43} + c_3c_{42}^2a_{43} +$
$2c_2c_{42}^2a_{43} + 2c_3a_{41}a_{43}^2 + 2c_3a_{42}a_{43}^2 + c_2a_{42}a_{43}^2 + c_2c_{42}^3 + c_3a_{43}^3)f^2f_xf_{yyy} + \frac{h^4}{4!}c_4^4f_{xxxx} + \frac{h^4}{3!}(c_4^3a_{41} + c_4^3a_{42} + c_4^3a_{43})ff_{xxxy}$
$+ \frac{h^4}{2!2!}c_4^2(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2f_{xxyy} + \frac{h^4}{3!}c_4(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 +$
$6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)\, f^3f_{xyyy} + \frac{h^4}{2!2!}(2c_2^2a_{41}a_{42} + 2c_3^2a_{41}a_{43} + 2c_3^2a_{42}a_{43} + c_2^2a_{42}^2 +$
$c_3^2a_{43}^2)ff_{xx}f_{yy.} + 0(h^5).$

$k_5 = f + h(a_{51} + a_{51} + a_{53} + a_{54})ffy + h^2(a_{21}a_{51} + a_{31}a_{53} + a_{32}a_{53} + a_{41}a_{54} +$
$a_{42}a_{54} + a_{43}a_{54})ff_y^2 + \frac{h^2}{2!}(a_{51}^2 + 2a_{51}a_{52} + 2a_{51}a_{53} + 2a_{51}a_{54} + a_{52}^2 + 2a_{52}a_{53} +$
$2a_{52}a_{54} + a_{53}^2 + 2a_{53}a_{54} + a_{54}^2)f^2f_{yy} + \frac{h^3}{2!}(a_{21}^2a_{52} + a_{31}^2a_{53} + 2a_{31}a_{32}a_{53} +$
$a_{32}^2a_{53} + a_{41}^2a_{54} + +2a_{41}a_{42}a_{54} + 2a_{41}a_{42}a_{54} + 2a_{41}a_{43}a_{54} + 2a_{42}a_{43}a_{54} +$
$a_{42}^2a_{54} + a_{43}^2a_{54} + 2a_{21}a_{51}a_{52} + 2a_{31}a_{51}a_{53} + 2a_{32}a_{51}a_{53} + 2a_{41}a_{51}a_{54} +$
$2a_{42}a_{51}a_{54} + 2a_{43}a_{51}a_{54} + 2a_{21}a_{52}^2 + 2a_{21}a_{52}a_{53} + 2a_{31}a_{52}a_{53} + 2a_{32}a_{52}a_{53} +$
$2a_{41}a_{52}a_{54} + 2a_{42}a_{52}a_{54} + 2a_{43}a_{52}a_{54} + 2a_{21}a_{52}a_{54} + 2a_{31}a_{53}^2 + 2a_{32}a_{53}^2 +$
$2a_{41}a_{53}a_{54} + 2a_{42}a_{53}a_{54} + 2a_{43}a_{53}a_{54} + 2a_{31}a_{53}a_{54} + 2a_{32}a_{53}a_{54} + 2a_{41}a_{54}^2 +$
$2a_{42}a_{54}^2 + 2a_{43}a_{54}^2)\, f^2f_yf_{yy} + h^3(a_{32}a_{21}a_{53} + a_{21}a_{42}a_{54} + a_{31}a_{43}a_{54} +$
$a_{32}a_{43}a_{54})ff_y^2 + \frac{h^3}{3!}(a_{51}^3 + 3a_{51}^2a_{52} + 3a_{51}^2a_{53} + 3a_{51}^2a_{54} + 3a_{51}a_{52}^2 +$
$6a_{51}a_{52}a_{53} + 6a_{51}a_{52}a_{54} + 3a_{51}a_{53}^2 + 6a_{51}a_{53}a_{54} + 3a_{51}a_{52}^2 + a_{52}^3 + 3a_{52}^2a_{53} +$
$3a_{52}^2a_{54}3a_{52}a_{53}^2 + 6a_{52}a_{53}a_{54} + 3a_{52}a_{54}^2 + a_{53}^3 + 3a_{53}^2a_{54} + 3a_{53}a_{54}^2 + a_{54}^3)f^3f_{yyy}hc_5f_x +$
$h^2(c_2a_{52} + c_3a_{53} + c_4a_{54})f_xf_y + \frac{h^2}{2!}c_5^2f_{xx} + h^2(c_5a_{51} +$
$c_5a_{52} + c_5a_{53} + c_5a_{54})ff_{xy} + \frac{h^3}{2!}(c_5^2a_{52} + c_3^2a_{53} + c_4^2a_{54})f_{xx}f_y + h^3(c_2a_{21}a_{52} +$
$c_3a_{31}a_{53} + c_3a_{32}a_{53} + a_4a_{41}a_{54} + a_4a_{42}a_{54} + a_4a_{43}a_{54})ff_{xy}f_y + h^3(a_{32}c_2a_{53} +$

Retrieval Number:100.1/ijam.A1144043123
DOI: 10.54105/ijam.A1144.043123
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication (LSP)

$c_2a_{42}a_{54} + c_3a_{43}a_{54})f_xf_y^2 + h^3(c_2c_5a_{52} + c_3c_5a_{53} + c_4c_5a_{54})f_xf_xy +$
$h^3(a_{21}c_5a_{52} + a_{31}c_5a_{53} + a_{32}c_{53} + a_{41}c_5a_{54} + a_{42}c_5a_{54} + a_{43}c_5a_{54})ff_yf_{xy} +$
$h^3(c_2a_{51}a_{52} + c_3a_{51}a_{53} + c_4a_{51}a_{54} + c_2a_{52}^2 + c_2a_{52}a_{53} + c_3a_{52}a_{53} + c_4a_{52}a_{54} +$
$c_2a_{52}a_{54} + c_3a_{53}^2 + c_4a_{53}a_{54} + c_3a_{54}a_{53} + c_4a_{54}^2)ff_xff_y + \frac{h^3}{3!}c_5^3f_{xxx} + \frac{h^3}{2!}(c_5^2a_{51} + c_5^2a_{52} + c_5^2a_{53} + c_5^2a_{54})ff_{xxy} +$
$\frac{h^3}{2!}(c_5a_{51}^2 + 2c_5a_{51}a_{52} + 2c_5a_{51}a_{53} + 2c_5a_{51}a_{54} + c_5a_{52}^2 + 2c_5a_{52}a_{53} + 2c_5a_{52}a_{54} + c_5a_{53}^2 + 2c_5a_{53}a_{54} +$
$c_5a_{54}^2)f^2f_{xyy} + 0(h^5)$. (3.4)

(Note: $c_2 = a_{21},\ c_3 = a_{31} + a_{32}, c_4 = a_{41} + a_{42} + a_{43,}c_5 = a_{51} + a_{52} + a_{53} + a_{54}$)

Resolving, we set $c_1 = 0$, $c_2 = ¼$, $c_3 = ¼$, $c_4 = ½$, $c_5 = 1$

Hence, the following equations are generated: $b_2 + b_3 + 2b_4 + 4b_5 = 2$

$3b_2 + 3b_3 + 12b_4 + 48b_5 = 16$ $b_2 + b_3 + 8b_4 + 64b_5 = 16$

Resolving the above, we have:

$b_1 = {}^1/_{6,} b_2 = {}^1/_{2,} b_3 = -{}^1/_{2,} b_4 = {}^2/_{3,} b_5 = {}^1/_{6,}$

Hence, we have the below equations:

$$-3a_{32} + 4a_{42} + 4a_{43} + a_{52} + a_{53} + 2a_{54} = 4$$
$$-3a_{32} + 8a_{42} + 8a_{43} + 4a_{52} + 4a_{53} + 8a_{54} = 12$$
$$-3a_{32} + 4a_{42} + 4a_{43} + a_{52} + a_{53} + 4a_{54} = 8$$
$$3a_{32}a_{43} + a_{32}a_{53} + a_{42}a_{54} + a_{43}a_{54} = 1$$

Setting $a_{32} = {}^1/_{2,} a_{42} = {}^1/_{4,} a_{43} = {}^1/_{4,}$ we have:

$2a_{52} + 2a_{53} + 2a_{54} = 2$
$2a_{52} + 2a_{53} + 8a_{54} = 15$
$8a_{52} + 8a_{53} + 16a_{54} = 19$
$a_{53} + a_{54} = 1$

Resolving, we have:

$a_{52} = {}^1/_{2,} a_{53} = -1, \quad a_{54} = 2$

Since, $c_2 = a_{21}, \therefore\ a_{21} = {}^1/_{4,} c_3 = a_{31} + a_{32} = {}^1/_{4,} a_{31} = -{}^1/_{4,} c_4 = a_{41} + a_{42} + a_{43} = {}^1/_{2,} a_{41} = 0, c_5 = a_{51} +$
$a_{52} + a_{53} + a_{54} = 1, \therefore\ a_{51} = -{}^1/_2$

Putting all the above parameters into the scheme, the fifth-stage fourth-order method becomes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 3k_2 - 3k_3 + 4k_4 + k_5)$$
$$k_1 = f(x_{n,}y_n)$$
$$k_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right)$$
$$k_3 = f(x_n + \frac{h}{4}, y_n + \frac{h}{4}(-k_1 + 2k_2))$$
$$k_4 = f(x_n + \frac{h}{2}, y_n + \frac{h}{4}(k_2 + k_3))$$
$$k_5 = f(x_n + h, y_n + \frac{h}{2}(-k_1 + k_2 - 2k_3 + 4k_4))$$

IV. IMPLEMENTATION OF THE FORMULA AND RESULTS

The formula is implemented on the initial – value problems below with the aid of FORTRAN programming language:

(i) $y^1 = -y,\ y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \frac{1}{e^{xn}}$

(ii) $y^1 = y,\ y(0) = 1, 0 \le x \le 1, \quad y(x_n) = e^{xn}$

(iii) $y^1 = 1 + y^2,\ y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \tan\left(x_n + {}^{\pi}/_4\right), h = 0.1$

Table 3.5 Tables of Results

(i) $y' = y - y^2,\ y(0) = 0.5,\ 0 \le x \le 1.$ Theoretical Solution is $y(x_n) = \frac{1}{1+e^{-x_n}}$

XN	TSOL	YN (5th stage)	Error (5th)	YN (Classical)	Error (class.)
.1D+00	0.5249791D+00	0.52497D+00	-.561D-08	0.52497D+00	0.13033D-08
.2D+00	0.5498340D+00	0.54983D+00	-.401D-07	0.54983D+00	0.26416D-08
.3D+00	0.5744425D+00	0.57444D+00	-.101D-06	0.57444D+00	0.40591D-08
.4D+00	0.5986876D+00	0.59868D+00	-.187D-06	0.59868D+00	0.55964D-08
.5D+00	0.6224593D+00	0.62245D+00	-.294D-06	0.62245D+00	0.72870D-08
.6D+00	0.6456563D+00	0.64565D+00	-.415D-06	0.64565D+00	0.91553D-08

.7D+00	0.6681877D+00	0.66818D+00	-.547D-06	0.66818D+00	0.11214D-07
.8D+00	0.6899744D+00	0.68997D+00	-.683D-06	0.68997D+00	0.13466D-07
.9D+00	0.7109495D+00	0.71095D+00	-.818D-06	0.71094D+00	0.15899D-07
.1D+01	0.7310585D+00	0.73105D+00	-.948D-06	0.73105D+00	0.18491D-07

(ii) $y' = -y, \ y(0) = 1, \ 0 \le x \le 1.$ Theoretical Solution is $y(x_n) = \frac{1}{e^{x_n}}, \ h = 0.1$

XN	TSOL	YN (5th stage)	Error (5thstage)	YN (Classical)	Error (classical)
.1D+00	0.90483D+00	0.90483D+00	0.222D-07	0.90483D+00	-.8196404044369D-07
.2D+00	0.81873D+00	0.81873D+00	0.401D-07	0.81873D+00	-.1483282683346D-06
.3D+00	0.74081D+00	0.74081D+00	0.545D-07	0.74081D+00	-.2013194597694D-06
.4D+00	0.67032D+00	0.67031D+00	0.657D-07	0.67032D+00	-.2428818514089D-06
.5D+00	0.60653D+00	0.60653D+00	0.744D-07	0.60653D+00	-.2747107467060D-06
.6D+00	0.54881D+00	0.54881D+00	0.807D-07	0.54881D+00	-.2982822888686D-06
.7D+00	0.49658D+00	0.49658D+00	0.852D-07	0.49658D+00	-.3148798197183D-06
.8D+00	0.44932D+00	0.44932D+00	0.882D-07	0.44932D+00	-.3256172068089D-06
.9D+00	0.40656D+00	0.40656D+00	0.897D-07	0.40656D+00	-.3314594766990D-06
.1D+01	0.36787D+00	0.36787D+00	0.902D-07	0.36787D+00	-.3332410563051D-06

(iii) $y' = y, \ y(0) = 1, \ 0 \le x \le 1.$ Theoretical Solution is $y(x_n) = e^{x_n}, \ h = 0.1$

XN	TSOL	YN (5th stage)	Error (5th stage)	YN (Classical)	Error (class.)
.1D+00	0.1105D+01	0.1105D+01	-.194D-07	0.1105D+01	0.847D-07
.2D+00	0.1221D+01	0.1221D+01	-.429D-07	0.1221D+01	0.187D-06
.3D+00	0.1349D+01	0.1349D+01	-.711D-07	0.1349D+01	0.310D-06
.4D+00	0.1491D+01	0.1491D+01	-.104D-06	0.1491D+01	0.457D-06
.5D+00	0.1648D+01	0.1648D+01	-.144D-06	0.1648D+01	0.632D-06
.6D+00	0.1822D+01	0.1822D+01	-.192D-06	0.1822D+01	0.838D-06
.7D+00	0.2013D+01	0.2013D+01	-.247D-06	0.2013D+01	0.108D-05
.8D+00	0.2225D+01	0.2225D+01	-.312D-06	0.2225D+01	0.136D-05
.9D+00	0.2459D+01	0.2459D+01	-.389D-06	0.2459D+01	0.169D-05
.1D+01	0.2718D+01	0.2718D+01	-.477D-06	0.2718D+01	0.208D-05

V. FINDINGS AND CONTRIBUTION TO KNOWLEDGE

This study reveals that f(x,y) functional derivatives can generate a formula that can improve the performance of results. After our implementation, it shows from the tables of numerical results that the method is highly efficient, because it compares favorably well with the existing classical fourth order explicit method.

VI. CONCLUSION

In conclusion, it is clear that when parameters from only f(x,y) functional derivatives are varied, there is high tendency that the method derived will be of higher performance than when all functional derivatives are carried out together. This will save time, energy and resources. Researchers in these areas do not need to go through rigorous procedures any longer in deriving new Runge-Kutta formula. The results generated from the MAPLE program also justifies this claim when compared with other existing methods.

DECLARATION STATEMENT

Funding	No, I did not receive.
Conflicts of Interest	No conflicts of interest to the best of my knowledge.
Ethical Approval and Consent to Participate	No, the article does not require ethical approval and consent to participate with evidence.
Availability of Data and Material	Not relevant.
Authors Contributions	I am only the sole author of the article.

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