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Proofs of Beal's Conjecture, Fermat's Conjecture, Collatz Conjecture and Goldbach Conjecture

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***Abstract:** In this article the elementary mathematical methods are used to prove Beal's Conjecture, Fermat's Conjecture, Collatz Conjecture and Goldbach Conjecture.*

***Keywords**: Beal's Conjecture, Fermat's Conjecture, Collatz Conjecture, Goldbach Conjecture.*

I. INTRODUCTION

The French mathematician Pierre de Fermat (1607-1665), conjectured that the equation $x^n + y^n = z^n$ has no solution in positive integers x, y and z if n is a positive integer ≥ 3[1]. The American Banker and amateur mathematician Mr. Daniel Andrew Beal formulated the Beal's conjecture in1993 [2] as a generalization of Fermat's Conjecture. Lothar Collatz introduced Collatz Conjecture in 1937[3,5]. It is also known as the 3n +1 problem . In 1742, the Russian mathematician Christian Goldbach introduced Goldbach Conjecture [4]. British Mathematician Andrew Wiles proved Fermat's Conjecture indirectly as a special case of modularity theorem for elliptic curves in 1995 [1] and so Fermat's Conjecture is also known as Fermat's Last Theorem. In this article these conjectures are proved directly using mathematical methods.

II. PRELIMINARIES

Statement 2.1: If $A^x + B^y = C^z$ where A,B,C,x,y and z are positive integers and x,y,z are greater than 2, then A,B and C must have a common prime factor.

Equivalently, the equation $A^x + B^y = C^z$ has no solutions in nonzero integers and pairwise coprime integers A,B,C if $x,y,z \geq 3$.

Statement 2.2: No three positive integers a, b, and c satisfy the equation $a^\alpha + b^\alpha = c^\alpha$ for any integer value of α greater than 2.

Definition 2.1. Hailstone sequence

Hailstone sequence corresponding to a positive integer n is a sequence $\{a_i\}$, i = 0,1,2, .., where a_i is obtained as the value applied to n recursively i times $a_i = f^i(n)$, $n \in \{1,2,3,4,\dots\}$ and i= 0,1,2,.. where $f^0(n) = n$ and for i > 0,

$$f^i(n) = \begin{cases} \frac{n}{2}, \text{if } n \text{ is even} \\ 3n+1, \text{if } n \text{ is odd} \end{cases}$$

Statement 2.3: For any positive integer $n \in$ N, the Hailstone sequence starting with n eventually ends in 1.

Definition 2.2. Prime gap

A prime gap is the difference between two successive prime numbers. The *n*-th prime gap, denoted g_n or $g(p_n)$ is the difference between the (*n* + 1)-th and the *n*-th prime numbers

i.e, $g_1 = 3-2 = 1$, $g_2 = 5-3 = 2$, $g_3 = 7-5 = 2$, $g_4 = 11-7 = 4$, $g_5 = 13-11 = 2$ and $g_6 = 17-13 = 4$

Definition 2.3. Prime gap interval

The ith prime gap interval is the set of positive integers y such that n^{th} prime number $\leq y \leq (n+1)^{th}$ prime number.

Examples: 1st prime gap interval is{2,3}, the 2nd prime gap interval is { 3,4,5}.

Statement 2.4.1: Every even number greater than 2 is sum of two prime numbers.

Statement 2.4.2: Every odd number greater than 7 is a sum of three odd prime numbers.

Statement 2.4.3: Every odd number greater than 7 is a sum of one prime number and an even number.

III. PROOF OF BEAL'S CONJECTURE

Basic results and notations that are used in the proof

1. If A^x is even then A is even.
2. If A^x is odd then A is odd.
3. Suppose $A^x + B^y = C^z$ where A,B,C,x,y and z are positive integers, then either all the three numbers A^x ,B^y , C^z must be even or any two of the numbers A^x ,B^y , C^z must be odd.
4. If all the three numbers A^x ,B^y , C^z are positive even,then the numbers A,B and C are even and they have a common prime factor 2.
5. Set of natural numbers is denoted by N. N = {1,2,3,..}
6. Set of Whole numbers is denoted by W. W={0,1,2,3,..}
7. A positive even number can be written as $2^u (2k+1)^v$ where k is non negative integer u,v $\in$ N
8. A positive odd number can be written as a product of $(2l_i+1)$ where $l_i \in$ W and $i \in$ N. In this representation the powers of same number is represented as having same numerical value to l_i but i takes distinct numbers.

Example 27= 3^3.

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27 = (2x1+1) (2x1+1)(2x1+1), Here $l_1 = l_2 = l_3 = 1$.
To prove Beal's Conjecture, it is enough to prove that if $A^x + B^y = C^z$ such that any two of the numbers A^x, B^y, C^z is odd, then there exists a common prime factor.

Statement 2.1: If $A^x + B^y = C^z$ where A,B,C,x,y and z are positive integers and x,y,z are greater than 2, then A,B and C must have a common prime factor.
Proof. Suppose $A^x + B^y = C^z$ where A,B,C,x,y and z are positive integers and x,y,z are greater than 2 then to prove that A,B and C must have a common prime factor.
Without loss of generality, suppose A^x is even and B^y, C^z are odd, to prove Beal's conjecture, it is enough to prove following lemma.

Lemma 3.1: If

$$2^{xu}(2k+1)^{xv} + \left[\prod_{i=1}^{m}(2l_i+1)\right]^y = \left[\prod_{j=1}^{n}(2m_j+1)\right]^z$$

then $(2l_i+1) = (2m_j+1)$ divides $(2k+1)^{xv}$ for some i and j where x, y, z > 2, l_i, m_j, k ∈ W and i,j, m,n, u,v ∈ N

Proof. Let $2^{xu}(2k+1)^{xv} + [\prod_{i=1}^{m}(2l_i+1)]^y = [\prod_{j=1}^{n}(2m_j+1)]^z \rightarrow$ (3.1).
Then $2^{xu}(2k+1)^{xv} = [\prod_{j=1}^{n}(2m_j+1)]^z - [\prod_{i=1}^{m}(2l_i+1)]^y \rightarrow$ (3.2).
Consider two integers *p* , *q* such that
$p \in \{l_i\,; i = 1,2,3,..m\}$ and
$q \in \{m_j\,; j = 1,2,3,..n\}$ in the equation (3.2).

Case 3.1.1. *p* = *q* = 0 for all *p, q*.
This case is a contradiction to equation (3.2).In the RHS of equation (3.2), $(2p+1) = (2q+1) = 1 \Longrightarrow$ RHS = 0 but LHS ≠ 0.

Case 3.1.2. There exists atleast one pair (*p*,*q*) such that
p = *q* ≠ 0.
In this case in RHS of equation (3.2), $(2p+1) = (2q+1)$ is a common factor. Since $(2k+1)^{xv}$ is the only odd factor in LHS of equation (3.2), $(2p+1) = (2q+1)$ must divide$(2k+1)^{xv}$.

Case 3.1.3. *p* , *q* > 0 and *p* ≠ *q* for all *p, q*. Suppose $l_i \neq m_j$ for all *i*, *j* where $\{l_i\,; i = 1,2,3,..m\}$ and $\{m_j\,; j = 1,2,3,..n\}$ in equation (3.2).Then $[\prod_{j=1}^{n}(2m_j+1)]^z$ and $[\prod_{i=1}^{m}(2l_i+1)]^y$ are odd relatively prime numbers.

Case 3.1.3.1.When *k* = 0. Let B $= \prod_{i=1}^{m}(2l_i+1)$ and C $= \prod_{j=1}^{n}(2m_j+1)$

It is trivial that if $2^x = C^z - B^y$ where B,C >1 and *x, y, z*> 2 are positive integers such that g.c.d (B,C) =1 , then the terms C^z and B^y are of the form $C^z = (r+1)2^x + t$ and
$B^y = r\,2^x + t$ where t =1,2,3,..,$2^x - 1$ and r ∈ W.

Suppose t is even, then C^z and B^y cannot be odd. Which is a contradiction to assumption. So it is enough to prove following lemma.

Lemma 3.2.There does not exist two odd numbers B ,C >1 such that $B^y = r\,2^x + t$ and $C^z = (r+1)\,2^x + t$ where x,y,z > 2, t = 1,3,5,…..,2^x -1 and r ∈W.
Suppose $r\,2^x + t = B^y$, B >1 is an odd number where r ∈ N, t = 1,3,5,….. ,2^x -1 , and *x*,*y* >2 . Since $C^z = (r+1)\,2^x + t = B^y + 2^x$, and gcd (B,C) =1, The possible values for C^z are 3^y, 5^y,…, $(B+2)^y$ where y ≥ 3.
We shall prove that, If B^y is an odd number then
$C^z \notin \{3^y, 5^y, \ldots, (B+2)^y\}$ for all y ≥ 3. using Principle of Mathematical Induction.

Step 1: For *n* = 1, *x* = 4, $B^y = 3^3$ = 27 =16 +11,
Here B =3, y=3 and t =11.
The choices for C^z are the set of numbers {33,35,37,39,41,43,45,47}. There are 8 odd numbers. Note that 5^3 =125 > 47 and 3^4 =81 > 47.It is clear that there does not exist an odd number C ∈ N such that for *z* >2, C^z ∈ {33,35,37,39,41,43,45,47}.Therefore there does not exist an odd number C ∈ N such that
$C^z = (r+1)\,2^x + t = B^y + 2^x = (B+2)^y$, for y = 3,4.

Step 2: Assume the result is true for y = p.
i.e, There does not exist an odd number C ∈ N such that $C^z = (B+2)^p$.
Now consider $(B+2)^{p+1} = (B+2)^p\,(B+2)$
If there exist an odd number C ∈N such that
$C^z = (B+2)^{p+1} = (B+2)^p\,(B+2)$ then $(B+2)^p$ is a factor of C^z. Therefore $(B+2)^p = C^s$ where s < z . Which is a contradiction to assumption that there does not exist an odd number C ∈ N such that $C^z = (B+2)^p$ since s < p. Hence the result is true for all y ≥ 3.In a similar way it can be proved the statement is true for all v, k, x, y, z ∈ N, r ∈ W where *x,y,z*> *2.*
Therefore $C^z \notin \{3^y, 5^y, \ldots, (B+2)^y\}$ for all y ≥ 3.Therefore there does not exist two odd numbers B ,C >1 such that $B^y = r\,2^x + t$ and $C^z = (r+1)\,2^x + t$ where x,y,z > 2, t = 1,3,5,…..,2^x -1 and r ∈W. Lemma 3.2 is a contradiction to equation (3.2).

Case 3.1.3.2. When *k* > 0 . Let B $= \prod_{i=1}^{m}(2l_i+1)$ and C $= \prod_{j=1}^{n}(2m_j+1)$.It is trivial that if $(2k+1)^{xv}\,2^x = C^z - B^y$ where B,C >1 and *x, y, z*> 2 are positive integers such that g.c.d (B,C) =1 , then the terms C^z and B^y are of the form $C^z = (r+1)\,(2k+1)^{xv}\,2^x + t$ and $B^y = r\,(2k+1)^{xv}\,2^x + t$ where t =1,2,3,.., $2^x\,(2k+1)^{xv}$-1 and v ∈ N , r ∈ W.
Suppose t is even, then C^z and B^y cannot be odd.
Which is a contradiction to assumption.
So it is enough to prove following lemma.

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Lemma 3.3. There does not exist two odd numbers B ,C >1 such that

$B^y = r\,(2k+1)^{xv}\,2^x + t$ and

$C^z = (r+1)\,(2k+1)^{xv}\,2^x + t$

where x,y,z > 2,

$t = 1,3,5,\ldots, 2^x\,(2k+1)^{xv}-1$

and v,k $\in$ N , r $\in$ W.

Suppose $r\,(2k+1)^{xv}\,2^x + t = B^y$,where B >1 is an odd number, $y > 2$, $t = 1,3,5,\ldots, 2^x\,(2k+1)^{xv}-1$ and v, k $\in$ N , r $\in$ W.

Since $C^z = (r+1)\,(2k+1)^{xv}\,2^x + t$, and gcd (B,C) =1, The possible values for C^z are $3^y, 5^y,\ldots, (B+2)^y$ where $y \geq 3$.We shall prove that If B^y is an odd number then $C^z \notin \{3^y, 5^y,\ldots, (B+2)^y\}$ for all $y \geq 3$ using Principle of Mathematical Induction.

Step 1: For $n = k = v =1$, $x=3$, $216 = 2^3 3^3$.

$B^y \in$ {217=216+1,219=216+3,221=216+5,…, 243=216+27, ….., 431=216+215}. Among these numbers $243 = 3^5$, therefore B = 3, t = 27 and y = 5 .The choices for C^z are 513,515,517,…,625, …,647. But 243= 216+ 27 and 512 +27= 539. There does not exist any odd number C such that C^z =539. Here $625 = 5^4$.But 243= 216+ 27 and 625 ≠ 539 . Note that $5^3 = 125 < 3^5$ and $5^4 = 625 > 3^5$.There fore , there does not exist an odd number C $\in$ N such that

$C^z = (B+2)^y$ for y = 3,4.

Step 2: Assume the result is true for y = p.

i.e, There does not exist an odd number C $\in$ N such that $C^z = (B+2)^p$ .Now consider $(B+2)^{p+1} = (B+2)^p\,(B+2)$ .If there exist an odd number C $\in$ N such that

$C^z = (B+2)^p\,(B+2)$, then $(B+2)^p$ is a factor of C^z.Therefore $(B+2)^p = C^s$ where s < z . Which is a contradiction to assumption that there does not exist an odd number C $\in$ N such that $C^z = (B+2)^p$.

Hence the result is true for all $y \geq 3$.In a similar way it can be proved the statement is true for all v ,k ,x ,y ,z $\in$ N ,r $\in$ W where *x,y,z > 2*. Therefore $C^z \notin \{3^y, 5^y,\ldots, (B+2)^y\}$ for all $y \geq 3$. There fore there does not exist two odd numbers B,C >1 such that $B^y = r\,(2k+1)^{xv}\,2^x + t$ and

$C^z = (r+1)\,(2k+1)^{xv}\,2^x + t$ where x,y,z > 2,

$t = 1,3,5,\ldots, 2^x\,(2k+1)^{xv}-1$ and v,k $\in$ N , r $\in$ W.

Lemma 3.3 is a contradiction to equation (3.2).

Therefore , If

$$2^{xu}(2k+1)^{xv} + \left[\prod_{i=1}^{m}(2l_i+1)\right]^y = \left[\prod_{j=1}^{n}(2m_j+1)\right]^z$$

then $(2l_i+1)=(2m_j+1)$ divides $(2k+1)^{xv}$ for some i and j where x, y, z > 2 , l_i , m_j , k $\in$ W and i,j, m,n, u,v $\in$ N

In a similar way the lemma 3.1 can be proved for the equation

$$\left[\prod_{i=1}^{m}(2l_i+1)\right]^x + \left[\prod_{j=1}^{n}(2m_j+1)\right]^y = 2^{zu}(2k+1)^{zv}$$

Hence the proof of Beal's Conjecture.

IV. PROOF OF FERMAT'S CONJECTURE

Statement 2.2: No three positive a, b, and c satisfy the equation $a^\alpha + b^\alpha = c^\alpha$ for any integer value of α greater than 2.

Proof. Beal's theorem implies that for any integers *a*,*b*and *c* if $a^\alpha + b^\alpha = c^\alpha$ then *a*,*b* and *c* must have a common prime factor. So cancelling the α^{th} power of common prime factor from both sides of the equation, without loss of generality suppose a^α , b^α and c^α are pairwise relatively prime numbers such that a^α is even b^α and c^α are odd. To prove Fermat's Conjecture, it is enough to prove that for any integer value of $\alpha > 2$, i,j,m n,u,v,w $\in$ N , l_i , m_j , k $\in$ W the following two equations cannot hold.

$$2^{wu}(2k+1)^{wv} + \left[\prod_{i=1}^{m}(2l_i+1)\right]^\alpha = \left[\prod_{j=1}^{n}(2m_j+1)\right]^\alpha$$

$$\left[\prod_{i=1}^{m}(2l_i+1)\right]^\alpha + \left[\prod_{j=1}^{n}(2m_j+1)\right]^\alpha = 2^{wu}(2k+1)^{wv}$$

Case 3.1.3.1 and case 3.1.3.2 restricting to $x = y = z = \alpha$ gives the proof for $k \geq 0$.

This proves the famous Fermat's Conjecture.

V. PROOF OF COLLATZ CONJECTURE

Statement 2.3: For any positive integer ,$n \in N$, the Hailstone sequence starting with n eventually ends in 1.

It is enough to prove that for all Hailstone sequences starting with any natural number *n*, there exists a natural number i such that there exists a term in the sequence $a_i = f^{\,i}(n) = 1$.

Theorem 5.1 .$\forall n \in N$, An exists,where *An* is the set that consists the numbers in Hailstone sequence starting with n.

Theorem 5.2. $Am \cap An \neq \emptyset, \forall m, n \in N$

Corollary5.3.$\bigcap_{n=1}^{\infty} An = A_2 \supset A_0 = \{1\}$, $n \in N$

Proof of Theorem 5.1. The set *An* consists the numbers a_i where a_i is obtained as the value applied to *n* recursively i times $a_i = f^{\,i}(n)$, $n \in N$.

As per definition, $f^0(n) = n$ and for $i > 0$

$$f^{\,i}(n) = \begin{cases} \frac{n}{2}, if\ \ n\ is\ even \\ 3n+1, if\ n\ is\ odd \end{cases}$$

It is clear that for every $n \in N$, $f^{\,i}(n)$is a natural number and so a_i exists. Hence An exists $\forall n \in N$.

Remark 2: The above proof never implies that *An* must contain 1 or *An* must be finite. The proof conveys that *An* exists and the elements in *An* , $n \in N$ are positive integers.

Theorem 5.2 . $Am \cap An \neq \emptyset, \forall m, n \in N$

To prove theorem 5.2, first we shall prove the following lemmas

Lemma 5.2.1: For any positive odd number $p > 1$, *the number* $(3p+1)$ *is even and* $(3p+1) > p$.

Proof. Trivial.

Since p *is odd*, $p = 2k+1$, where $k \in N$,

$p < (3p+1) = 3(2k+1)+1 = 6k+4 = 2(3k+2)$

Lemma 5.2.2: *For any positive odd number* $p > 1$, *If* $\frac{(3p+1)}{2}$ *is odd then* $\frac{(3p+1)}{2} > p$

Proof. Trivial

$$\frac{(3p+1)}{2} = 1.5p + 0.5 > p$$

Lemma 5.2.3: For any positive odd number $p > 1$, *If* $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Proof.

Since p *is odd*, $p = 2k+1$, where $k \in N$, *and* $\frac{(3p+1)}{2} = (3k+2)$

It is obvious that $1.5k < 2k$ *for all* $k \in N$.

i.e, $(1.5k+1) < (2k+1)$, *where* $k \in N$

i.e, $\frac{(3k+2)}{2} < (2k+1)$, *where* $k \in N$

i.e, $\frac{(3k+2)}{2^i} < (2k+1) = p$, *where* $i, k \in N$

i.e, $\frac{(3k+2)}{2^{i-1}} < (2k+1) = p$, *where* $i > 1, k \in N$

i.e, $\frac{(3p+1)}{2^i} = \frac{(3k+2)}{2^{i-1}} < p$, *where* $i > 1, k \in N$

i.e, If $\frac{(3p+1)}{2^i}$ *is odd* or even *then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Hence, If $\frac{(3p+1)}{2^i}$ *is odd then* $\frac{(3p+1)}{2^i} < p$ *where* $i > 1$

Remark: 1. The Lemma 5.2.3 also holds if $\frac{3p+1}{2^i}$ is even.

Remark: 2. When $p = 1$, $(3p+1)/(2^2) = 1 = p$

Corollary 5.1: From the above proofs and the definitions of $f^i(n)$ and An, we shall observe the following inequalities and sub set relations.

If $p > 1$ is an odd number

5.2.3.1 $A_{3p+1}\ C\ A_p$

5.2.3.2

$$\text{If } \frac{(3p+1)}{2} \text{ is odd then } \frac{(3p+1)}{2} > p \text{ and } A_{\frac{(3p+1)}{2}} \subset A_{(3p+1)} \subset A_p$$

5.2.3.3

$$\text{If } \frac{(3p+1)}{2^i} \text{ is odd then } \frac{(3p+1)}{2^i} < p \text{ where } i > 1 \text{ and}$$

$$A_{\frac{(3p+1)}{2^i}} \subset A_{\frac{(3p+1)}{2^{i-1}}} \subset \ldots \subset A_{\frac{(3p+1)}{2}} \subset A_{3p+1} \subset A_p$$

Let $p > 1$ be any odd numbers in N, then the relations 5.2.3.2 and 5.2.3.3 imply that there exist some odd number q holding any of the following inequalities.

(i) $\quad q = \frac{(3p+1)}{2} > p$

(ii) $\quad p = \frac{(3q+1)}{2} > q$

(iii) $\quad q = \frac{(3p+1)}{2^i} < p$

(iv) $\quad p = \frac{(3q+1)}{2^i} < q$

Corollary 5.2:

Let p be any positive odd number .Then atleast any one of the following cases will hold.

Case 1: There exist some odd number q such that $Ap \subset Aq$.

Case 2: There exist some odd number q such that $Aq \subset Ap$.

Case 3: There exists an even number k such that $A_k \subset Ap$.

Define a relation R on the set $\{An\}$, where $n \in N$ such that $Ap\ R\ Aq$ iff $Ap \subseteq Aq$. Now R defines a partial order relation since it is reflexive, anti symmetric and transitive. Now($\{An\}$, R) is a partially ordered set.

Lemma 5.2.3.1: The minimum element in a partially ordered set is unique.

Proof: Suppose there are two minimum elements Ap and Aq. Since Ap is minimum $Ap \subseteq Aq$ .Since Aq is also minimum $Aq \subseteq Ap$. Hence $Ap = Aq$. That means the minimum element is unique.

Lemma 5.2.3.2: A_5 is unique minimum element in partially ordered set ($\{An\}$, R) for a set of odd numbers (say P).

Proof: From the relation R ,definition of An , lemmas 5.2.2 to 5.2.3.1 ,corollary 1 and corollary 2 we get

Observation 1: By corollary 1, when p=3, we get $A_5 R A_3$.

Observation 2: The relation R ,definition of An , lemmas 5. 2.2 to 5.2.3.1 , corollary 1 and corollary 2, when applied to odd numbers, we get $A_5 R\ A_{13}\ R\ A_{17}\ R\ A_{11}\ R\ A_7\ R\ A_9\ R$ …..

Observations 1 , observation 2 and lemma 5.2.3.1 implies that A_5 is the unique minimum element in partially ordered set ($\{An\}$, R) for a set of odd numbers .Let P be that set of odd numbers in N for which A_5 is unique minimum element.

Then $\bigcap_{p \in P}^{\infty} Ap = A_5$ →Equation (5.1).

Let Q be the set of odd numbers in the set $N - P$.i.e, Q ={ x/ x is an odd number in N-P}

Lemma 5.2.4: If m is a positive even number then it is a term of either the sequence{ 2^u} or the sequence{$(2k+1)^v\ 2^u$} where u,v,k $\in N$.

Proof. The first sequence { 2^u} contains all even numbers that can be written as 2^u . Suppose m is an even number such that $m \neq 2^u$.Then $m = 2s$ where $s > 1$ and s is a natural number. If s is odd, then m is a term of the second sequence{$(2k+1)^v 2^u$}.

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If s is even, s can be written as product of powers of prime numbers . Since all prime numbers except 2 are odd, one factor of s is of the form (2k+1) , k $\in N$. Hence $m = 2s$ is a term of the second sequence $\{(2k+1)^v 2^u\}$. Hence If m is an even number then it is a term of either the sequence $\{2^u\}$ or the sequence $\{(2k+1)^v 2^u\}$ where u,v, k $\in N$.

Let S $=\{x/ x\in\{2^u\}$ or $x\in\{(2k+1)^v 2^u\}\}$. Now $N = SUQUP$. The sets P and SUQ is a partition for N.

Lemma 5.2.4.1: A_2 is unique minimum element in partially ordered set $(\{An\}, R))$ where $n \in SUQ$.

Proof : The definition of An and relation R implies that A_2 is included in all An where $n\in\{x / x = 2^u, u\in N\}$.Also $A_{(2k+1)^v}\ R\ A_{(2k+1)^v 2^u}$ where u,v, k $\in N$.If $(2k+1)^v = p \in P$, then by lemma 5. 2.3.2, $A_5 R A_{(2k+1)^v}$ and by lemma 5.2.3 , $A_2 R A_5$ Hence $A_2 R A_{(2k+1)^v 2^u}$. Suppose $(2k+1)^v = p \in Q$. By lemma 5.2.3, $A_4\ R\ A_1$, and $A_2\ R\ A_1$. For all odd $p \in Q$ where $p > 1$, by corollary 5.1 and corollary 5.2, there exists an odd q such that either $AqRApor\ Ap\ R\ Aq$. If $q \in P, A_2 R A_5 RAq$ .Hence Ap includesA_2 .If $q \in Q$, without loss of generality suppose $A_r, r \in Q$ be the set such that $A_r = \cap A_{qi}$ for a set of $q_i \in Q$ where $I = 1,2,3,\ldots$. Now$3r+1$ is even and except 1 there is no $q_i \in Q$ such that $A_{qi}RA_{3r+1}$. Hence by corollary 5.1, Subset relation 5.2.3.3 we get $3r+1\in\{x / x = 2^u, u\in N\}$.For all odd numbers in q in Q , the number $3q+1$ is even and belongs to S. This implies that A_2 is the unique minimum element in partially ordered set $(\{An\}, R)$ where $n \in SUQ$.

Hence $\bigcap_{q\in SUQ}^{\infty} Aq = A_2$ →Equation (5.2)

From equations (5.1) and (5.2), $\forall m,n \in N = SUPUQ, Am \cap An \supset A_5 \cap A_2 = A_2$. Hence $\forall m,n \in N,\ Am \cap An \neq \emptyset$.

Corollary 5.3. $\bigcap_{n=1}^{\infty} An = A_2 \supset A_0 = \{1\}, n \in N$.

Proof. $\bigcap_{n=1}^{\infty} An = \bigcap_{p\in P}^{\infty} Ap \bigcap_{q\in SUQ}^{\infty} Aq = A_5 \cap A_2 = A_2 \supset A_0$.

This shows that the set A_0 is subset of all $, n \in N$. Which implies that the element 1 belongs to all Hailstone sequences. Therefore, for all Hailstone sequences staring with $n, n \in N$, there exists a number i n N such that $a_i = f^i(n) = 1$. In other words all the Hailstone sequences staring with $n, n \in N$, contains the term 1. This proves the famous Collatz Conjecture.

VI. PROOF OF GOLDBACH CONJECTURE

Statement 2.4.1: Every even number greater than 2 is sum of two prime numbers**.**

Proof. Let $n > 1$ be a positive integer. Let $E_n = \{e / e$ is an even number $\leq n\}$ .Let $P_n = \{p_1 = 2, p_2, p_3, \ldots, p_s\}$ be the set of all prime numbers $\leq n$ .It is enough to prove that every even number $e \in E_n$ where $e \neq 2$ can be written as $p_i + p_j$ where $i , j = 1,2,3,\ldots s$.

Define

$M_n = \{e_{i,j} / e_{i,j} = p_i + p_j = 2\left[\frac{p_i - 1}{2} + \frac{p_j - 1}{2} + 1\right]$ where $i, j = 1,\ldots,s.\}$

To prove that $E_n \subset M_n \cup \{2\}$ for all $n \in N - \{1\}$.

Let $e_{i,j} \in E_n$ where $e_{i,j} \neq 2$ .To prove that $e_{i,j} \in M_n$, it is enough to prove the following lemmas.

Lemma 6.1: Corresponding to each positive integer $x \leq (n/2)$ there exists $p_i, p_j \in P_n$.such that $2(x+1) = p_i + p_j$

Proof. Since every prime number except 2 are odd numbers ,$p_i = 2l+1$and $p_j = 2m+1$for some positive integers l , m . There fore $p_i + p_j = 2l+1 + 2m+1 = 2(l+m+1) = 2(x+1)$ where $x = l+m$.

Now we shall prove that for all even numbers $4 \leq 2(x+1) \leq n$, there exists p_i , $p_j \in$ P where $l = (p_i - 1)/2$, $m = (p_j - 1)/2$ and $x = l+m \leq (n/2)$

Let g_i be the i^{th} prime gap and let $g_{i,1}, g_{i,2}, \ldots, g_{i,r}$, be the positive integers in ascending order in the i^{th} prime gap interval where $g_{i,1} = p_i$ and $g_{i,r} = p_{i+1} \leq n$.

For each $i \in$ N and corresponding prime gap g_i , define the two neighbourhood sets N_i^+ and N_i^- , such that $N_i^+ = \{g_{i1} = p_i, p_{i+1}, \ldots, p_u\} \cap P_n$ where $p_i, p_{i+1}, \ldots, p_u$ are $p_{i+1} - p_i$ number of consecutive prime numbers and $N_i^- = \{g_{i1} = p_i, p_{i-1}, \ldots, p_v\} \cap P_n$ where $p_i, p_{i-1}, \ldots, p_v$ are $p_i - p_{i-1}$ number of consecutive prime numbers

It is obvious that there are $p_{i+1} - p_i$ distinct positive integers in $[p_i, p_{i+1})$ where $p_i, p_{i+1} \in P_n$ .Selecting $p_{i+1} - p_i$ number of consecutive prime numbers greater than or equal to $p_i \in N_i^+$,it is possible to get $\frac{(p_{i+1} - p_i)(p_{i+1} - p_i + 1)}{2} \geq (p_{i+1} - p_i)$ distinct positive integers $x = l + m \leq \frac{n}{2}$,where $l = \frac{p_i - 1}{2}$, $m = \frac{p_j - 1}{2}$ such that $p_i, p_{i+1} \in P_n$ .Among these distinct positive integers$p_{i+1} - p_i$number of positive integers x must be such that

$(p_i - 1) \leq x \leq (p_\alpha - 1)$

since

$\frac{p_i - 1}{2} + \frac{p_i - 1}{2} = p_i - 1$ and

$\frac{p_\alpha - 1}{2} + \frac{p_\alpha - 1}{2} = p_\alpha - 1$.

Similarly, selecting $p_i - p_{i-1}$ number of consecutive prime numbers less than or equal to $p_i \in N_i^-$, it is possible to get

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$\frac{(p_i - p_{i-1})(p_i - p_{i-1}+1)}{2} \geq (p_i - p_{i-1})$ distinct positive integers $x = l + m \leq \frac{n}{2}$,where $l = \frac{p_i - 1}{2}$, $m = \frac{p_j - 1}{2}$ such that $p_i, p_j \in P_n$.
Among these distinct positive integers $p_i - p_{i-1}$ number of positive integers must be such that
$(p_\beta - 1) \leq x \leq (p_i - 1)$
since
since $\frac{p_i - 1}{2} + \frac{p_i - 1}{2} = p_i - 1$ and
$\frac{p_\beta - 1}{2} + \frac{p_\beta - 1}{2} = p_\beta - 1$.
Let
$A = \{x / (p_{i-1} - 1) \leq x \leq (p_i - 1), i = 2,3,\ldots,\beta,\ldots,\alpha\}$.

Now, It is obvious that $\frac{p_\alpha - 1}{2} < |A| \leq p_\alpha - 1$.Suppose $\mu < p_\alpha - 1$ be a positive integer in the α^{th} prime gap interval such that $\mu \notin A$.
Note that $\mu = a_1 + c_1$ for some positive integers $a_1, c_1 \in \{1,2,3,\ldots, \mu - 1\}$.

Lemma 6.2: If $\mu < p_\alpha - 1$ is an integer in the α^{th} prime gap interval then there must exist two positive integers $x_1, y_1 \in A = \{x / (p_{i-1} - 1) \leq x \leq (p_i - 1), i = 2,3,\ldots,\beta,\ldots,\alpha\}$
such that $x_1 = a_1 + b_1$ and $y_1 = c_1 + d_1$
where a_1, b_1, c_1, d_1 are of the form $\frac{p_t - 1}{2}$, $t = 1,\ldots,\alpha$.

Proof. Suppose there does not exist $x_1 = a_1 + b_1$ and $y_1 = c_1 + d_1$
where a_1, b_1, c_1, d_1 are of the form $\frac{p_t - 1}{2}$, $t = 1,\ldots,\alpha$.Then $\mu, x_1, y_1 \notin A$. Since x_1, y_1 are positive integers in the $(\alpha - j)^{th}$ prime gap interval , where j= 0,1,2,3,..α-1 ,there must exist $a_2, b_2, c_2, d_2 \in \{1,2,3,\ldots, \mu - 1\}$ and $x_2, y_2 \in A$ such that $x_2 = a_2 + b_2$ and $y_2 = c_2 + d_2$
where a_2, b_2, c_2, d_2 are of the form $\frac{p_t - 1}{2}$, $t = 1,\ldots,\alpha$.
Suppose there does not exist $x_2 = a_2 + b_2$ and $y_2 = c_2 + d_2$, then $\mu, x_1, y_1, x_2, y_2 \notin A$. Continuing this argument, we get a set of $\frac{p_\alpha - 1}{2}$ numbers that doest not belong to A . Since $\frac{p_\alpha - 1}{2} < |A| \leq p_\alpha - 1$., the argument leads to a contradiction. Therefore by method of infinite descent, there must exist two positive integers
$x_1 = a_1 + b_1$ and $y_1 = c_1 + d_1$ where
a_1, b_1, c_1, d_1 are of the form $\frac{p_t - 1}{2}$, $t = 1,\ldots,\alpha$.
This implies that $\mu = a_1 + c_1 = \frac{p_i - 1}{2} + \frac{p_j - 1}{2}$,
$i, j = 1,\ldots,\alpha$.Therefore , corresponding to each positive integer $x \leq \frac{n}{2}$,there exists $p_i, p_j \in P_n$ such that $2(x+1) = p_i + p_j$. Now $e_{i,j} \in E_n$ where $e_{i,j} \neq 2$ and $i \neq j$, implies that $e_{i,j}$ is an even number. i.e, $e_{i,j} = 2k$ where k is any positive integer such that $k \leq \frac{n}{2}$.
Applying lemma 6.1, there exists $p_i, p_j \in P_n$ such that $2k = p_i + p_j$.
Now
$p_i + p_j = 2\left[\frac{p_i - 1}{2} + \frac{p_j - 1}{2} + 1\right]$.
Therefore $e_{i,j} \in M_n$
Therefore $E_n \subset M_n \cup \{2\}$ for all n $\in$ N$-\{1\}$.

Statement 2.4.2: Every odd number greater than 7 is a sum of three odd prime numbers.
Proof . The unit digit of every even number can be any of the number in {0,2,4,6,8}. If the prime numbers 3,5,7 or 11 is added to every even number then the digit in the unit place of sum will be 1,3,5,7 or 9.Therefore every odd number can be obtained by adding 3,5,7or 11 with an even number. The statement 1 implies that every even number greater than 2 is sum of two prime numbers**.** Therefore every odd number greater than 7 is a sum of three odd prime numbers**.**
Statement 2.4.3: Every odd number greater than 7 is a sum of one prime number and an even number.
Proof. The statement 2.4.2 implies that every odd number greater than 7 is a sum of three odd prime numbers**.** It is obvious that sum of two odd numbers is always even. Therefore considering the sum of two odd primes as an even number statement 2.4.2 implies statement 2.4.3.

Illustrative Example:
When $n = 24$.
$E_{24} = \{2,4,6,8,10,12,14,16,18,20,22,24\}$
$P_{24} = \{p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, p_8 = 19, p_9 = p_s = 23\}$
$M_{24} = \{4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34, 36, 38, 40, 42,46\}$.
$g_1 = 3 - 2 = 1$, $g_2 = 5 - 3 = 2$, $g_3 = 7 - 5 = 2$, $g_4 = 11 - 7 = 4, g_5 = 13 - 11 = 2, g_6 = 17 - 13 = 4,$
$g_7 = 19 - 17 = 2, g_8 = 23 - 19 = 4$

Consider $g_2 = 5 - 3 = 2$. The corresponding prime gap interval is{ 3,4,5}.

i.e, $g_{2,1} = p_i = 3, g_{2,2} = 4, g_{2,3} = p_j = 5$.

Selecting $5 - 3 = 2$ prime numbers ≥ 3, we can form $N_2^+ = \{3,5\}$.

Selecting $3 - 2 = 1$ prime numbers ≤ 3, we can form $N_2^- = \{3\}$

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There are 2 elements in N_2^+ ,therefore $\frac{2(3)}{2} = 3$ distinct positive integers $3 - 1 = 2 \le x \le 4 = 5 - 1$ can be formed such that $x = l + m$ where $l = \frac{p_i - 1}{2}$, $m = \frac{p_j - 1}{2}$.

Note that $2 = \frac{3-1}{2} + \frac{3-1}{2}$, $3 = \frac{3-1}{2} + \frac{5-1}{2}$, $4 = \frac{5-1}{2} + \frac{5-1}{2}$

The corresponding even numbers in E_n are 2(2+1) = 6,

2(3+1) = 8 and 2(4+1) =10

Now consider $g_3 = 7 - 5 = 2$. The corresponding prime gap interval is{ 5,6,7}.

Selecting $7 - 5 = 2$ prime numbers $\ge$ 5, we can form $N_3^+ = \{5,7\}$.

Selecting $5 - 3 = 2$ prime numbers $\le$ 5, we can form $N_3^- = \{3,5\}$

There are 2 elements in N_3^+ ,therefore $\frac{2(3)}{2} = 3$ distinct positive integers (5 -1)= 4 $\le x \le$ 6= (7-1) can be formed such that $x = l + m$ where $l = \frac{p_i - 1}{2}$, $m = \frac{p_j - 1}{2}$.

Note that $4 = \frac{5-1}{2} + \frac{5-1}{2}$, $5 = \frac{5-1}{2} + \frac{7-1}{2}$ and $6 = \frac{7-1}{2} + \frac{7-1}{2}$. The corresponding even numbers in E_n are 2(4+1) = 10, 2(5+1) = 12 and 2(6+1) =14.

Now consider $g_4 = 11 - 7 = 4$. The corresponding prime gap interval is{ 7,8,9,10,11}.

Selecting $11 - 7 = 4$ prime numbers $\ge$ 7, we can form $N_4^+ = \{7,11,13,17\}$.

Selecting $7 - 5 = 2$ prime numbers $\le$ 7, we can form $N_4^- = \{5,7\}$.There are 4 elements in N_4^+, therefore $\frac{4(5)}{2} = 10$ distinct positive integers (7-1) = 6 $\le x \le$ 16= (17-1) can be formed such that $x = l + m$ where $l = \frac{p_i - 1}{2}$, $m = \frac{p_j - 1}{2}$. Among these, it is enough to get the numbers between $6 \le x \le 10 = (11 - 1)$

Note that $6 = \frac{7-1}{2} + \frac{7-1}{2}$, $8 = \frac{7-1}{2} + \frac{11-1}{2}$, $9 = \frac{7-1}{2} + \frac{13-1}{2}$, $10 = \frac{11-1}{2} + \frac{11-1}{2}$.

Note that there exists positive integers a_1 = 2 and c_1=5 such that $\mu = a_1 + c_1 = 2 + 5 = 7$

Let A = {3,4,5,6,8,9,10} . In this set x_1= 6 = 2 + 4 and y_1 = 8 = 5 + 3 where $2 = \frac{5-1}{2}$ and $5 = \frac{11-1}{2}$.

Therefore $7 = \frac{5-1}{2} + \frac{11-1}{2}$ .The corresponding even numbers in E_n are 2(6+1) = 14, 2(7+1) = 16 ,

2(8+1) =18, 2(9+1)=20 and 2(10+1)=22.

Consider $g_5 = 13 - 11 = 2$. N_5^+= {11,13}.

$10 = \frac{11-1}{2} + \frac{11-1}{2}$,$11 = \frac{11-1}{2} + \frac{13-1}{2}$,$12 = \frac{13-1}{2} + \frac{13-1}{2}$.

The corresponding even numbers in E_n are 2(10+1) = 22, 2(11+1) = 24. Therefore $E_{24} \subset M_{24} \cup \{2\}$.

Hence the proof of famous Goldbach conjecture.

VII. CONCLUSION

In this article the famous unproven conjectures in Number theory Beal's, Collatz and Goldbach's are proved using elementary methods. Fermat's conjecture is already proved by Andrew Wiles in indirect method. Since Beal's Conjecture is generalization of Fermat's Conjecture, the proof of Fermat's Conjecture that is discussed in this article as deduction from the proof of Beal's is the direct proof. Fermat's equation has solutions in non integers. Interpreting those solutions as measures of acceptance and rejections of an alternative in a network in comparison with other alternatives, study on Fermat's Fuzzy Graphs and its applications in decision problems is under progress. The model and applications of Beal's Fuzzy Graphs, Applications of Collatz and Goldbach's theorems are also under investigation.

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