Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-2 Issue-1, April 2022

On the Integer Solution of the Transcendental Equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$

Sriram S, Veeramallan P

***Abstract:** Let C be a positive non-square integer. In this paper, we look at the complete solutions of the Transcendental equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$, where $x^2 - Cy^2 = \alpha^2$ or 2^{2t}. In addition, we find repeated relationships in the solutions to this figure.*

***Keywords:** Transcendental equation, Pell equation, Diophantine equations*

AMS Subject Classification: 11J70, 11J81, 11D45, 11A55, 11Y65

I. INTRODUCTION

The Transcendental equation plays an important role in solving various Science and Engineering problems. Here is a Transcendental equation, which can be solved by standard calculation methods. As such, this paper offers a novel view of solving Transcendental equations using the concept of Pell equations. Let $C \neq 1$ be a positive non-square integer and N be any fixed positive integer. After that the figure

$$x^2 - Cy^2 = \pm N \quad (1)$$

is known as the Pell equation and is named after John Pell (1611-1685), a mathematical who sought complete solutions to such calculations in the seventeenth century. In N=1, the Pell equation

$$x^2 - Cy^2 = \pm 1 \quad (2)$$

is known as the classical Pell equation and was first studied by Brahmagupta (598-670) and Bhaskara (1114-1185), see [1]. Pell equation $x^2 - Cy^2 = 1$ was solved by Lagrange according to simple continued fractions. Lagrange was the first to prove that $x^2 - Cy^2 = 1$ has innumerable solutions in integers if $C \neq 1$ is a whole number that is not a square. The first minimal solution for the whole number(x_1, y_1) of this calculation is called the basic solution because all other solutions can be found in it. If (x_1, y_1) is the basic solution of $x^2 - Cy^2 = 1$, then a good n-thsolution(x_n, y_n) is defined

Manuscript received on 08 March 2022 | Revised Manuscript received on 20 March 2022 | Manuscript Accepted on 15 April 2022 | Manuscript published on 30 April 2022.

* Correspondence Author

Sriram S*, Assistant Professor, National College (Affiliated to Bharathidasan University), Tiruchirappalli, Tamilnadu, India. E-mail: sriram.priya02@yahoo.com

Veeramallan P, P.G Assistant, Department of Mathematics, GHSS, Yethapur, Salem-636 117, Tamilnadu, (Part-time Research Scholar, National College (Affiliated to Bharathidasan University), Tiruchirappalli). Email: veeramallan1986@gmail.com

$$x_n + y_n\sqrt{C} = \left(x_1 + y_1\sqrt{C}\right)^n \quad (3)$$

Allow $[a_0; \overline{a_1, a_2, \dots, a_r, 2a_0}]$ became a simple continued fraction of $\sqrt{C}$, where $a_0 = \lfloor\sqrt{C}\rfloor$. Allow $p_0 = a_0$, $p_1 = 1 + a_0a_1$, $q_0 = 1$, $q_1 = a_1$. Usually,

$$p_n = a_np_{n-1} + p_{n-2} \text{ and } q_n = a_nq_{n-1} + q_{n-2} \quad (4)$$

for $n \geq 2$. Then the basic solution of $x^2 - Cy^2 = 1$ is

$$(x_1, y_1) = \begin{cases} (p_r, q_r) & if\ r\ is\ odd \\ (p_{2r+1}, q_{2r+1}) & if\ r\ is\ even \end{cases} \quad (5)$$

On the other hand, in the case of (1) and (2), it is known that if (f_1, g_1) and (x_{n-1}, y_{n-1}) are complete solution of $x^2 - Cy^2 = \pm N$ and $x^2 - Cy^2 = 1$, respectively, then $(f_n, g_n)$and is the solution of $x^2 - Cy^2 = \pm N$, where

$$f_n + g_n\sqrt{C} = \left(x_{n-1} + y_{n-1}\sqrt{C}\right)\left(f_1 + g_1\sqrt{C}\right) \quad (6)$$

for $n \geq 2$.

II. MATERIALS AND METHODS

In this function, we look at the transcendental equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$.

Separating both sides and simplifying, we have

$$z = x + 2 \pm \sqrt{x^2 - Cy^2} \quad (7)$$

Take $x^2 - Cy^2 = \alpha^2$, so that $z = x + 2 \pm \alpha$. After that we can give the following theorem.

Theorem: 1

Let $(x_1, y_1)$to be the basic solutions for Pell equation $x^2 - Cy^2 = \alpha^2$ and allow

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} x_1 & Cy_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (8)$$

for $n \geq 2$. Then the complete solutions of the transcendental equation $z = x + 2 \pm \sqrt{x^2 - Cy^2}$, $(x^2 - Cy^2 = \alpha^2)$ are (x_n, y_n, z_n), where

$$(x_n, y_n, z_n) = \left(\frac{f_n}{\alpha^{n-1}}, \frac{g_n}{\alpha^{n-1}}, \frac{f_n}{\alpha^{n-1}} + (2 \pm \alpha)\right) \quad (9)$$

Proof. We validate the theorem using mathematical input method. In $n = 1$, we come from (8), $(f_1, g_1) = (x_1, y_1)$ which is the basic solution for$x^2 - Cy^2 = \alpha^2$. Now we assume that the pell equation $x^2 - Cy^2 = \alpha^2$ satisfied with(x_{n-1}, y_{n-1}). That is,

Retrieval Number:100.1/ijam.C1120041322
DOI: 10.54105/ijam.C1120.041322
Journal Website: www.ijam.latticescipub.com

Published By:
Lattice Science Publication

$$x_{n-1}^2 - Cy_{n-1}^2 = \frac{f_{n-1}^2 - g_{n-1}^2}{\alpha^{2n-4}} = \alpha^2 \quad (10)$$

and indicates that it contains (x_n, y_n).

Indeed in (8), it is easily proved that,

$$f_n = x_1 f_{n-1} + Cy_1 g_{n-1} \quad (11)$$

$$g_n = y_1 f_{n-1} + x_1 g_{n-1} \quad (12)$$

Hence,

$$x_n^2 - Cy_n^2 = \frac{f_n^2 - g_n^2}{\alpha^{2n-2}}$$
$$= \frac{(x_1 f_{n-1} + Cy_1 g_{n-1})^2 - C(y_1 f_{n-1} + x_1 g_{n-1})^2}{\alpha^{2n-2}}$$
$$= \frac{x_1^2(f_{n-1}^2 - Cg_{n-1}^2) - Cy_1^2(f_{n-1}^2 - Cg_{n-1}^2)}{\alpha^{2n-2}}$$
$$= \frac{x_1^2 - Cy_1^2}{\alpha^{2n-2}}(f_{n-1}^2 - Cg_{n-1}^2)$$

Applying (10), it is easily seen that,

$$f_{n-1}^2 - Cg_{n-1}^2 = \alpha^{2n-4}\alpha^2 = \alpha^{2n-2}$$

Hence, we conclude that,

$$x_n^2 - Cy_n^2 = x_1^2 - Cy_1^2 = \alpha^2$$

Therefore, (x_n, y_n) is also a solution of the Pell equation $x^2 - Cy^2 = \alpha^2$. Since n is arbitrary, we get all integer solutions of the pell equation $x^2 - Cy^2 = \alpha^2$.
Since $z_n = x_n + 2 \pm \alpha$, so that $z_n = \frac{f_n}{\alpha^{n-1}} + (2 \pm \alpha)$. Therefore, (x_n, y_n, z_n) be the solution of the equation (7) for which $x^2 - Cy^2 = \alpha^2$.

Corollary: 2

Let (x_1, y_1) be the basic solution of the Pell equation $x^2 - Cy^2 = \alpha^2$, and then

$$x_n = \frac{x_1 x_{n-1} + Cy_1 y_{n-1}}{\alpha} \quad (13)$$

$$y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{\alpha} \quad (14)$$

Therefore $z_n = \frac{x_1 x_{n-1} + Cy_1 y_{n-1}}{\alpha} + (2 \pm \alpha) \quad (15)$

Also, $\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -\alpha y_1 \quad (16)$

Proof. In (8), we have $f_n = x_1 f_{n-1} + Cy_1 g_{n-1}$ and $g_n = y_1 f_{n-1} + x_1 g_{n-1}$.

In (9), we have $f_n = \alpha^{n-1} x_n$ and $g_n = \alpha^{n-1} y_n$.

Therefore, $\alpha^{n-1} x_n = x_1 \alpha^{n-2} x_{n-1} + Cy_1 \alpha^{n-2} y_{n-1}$

$$x_n = \frac{x_1 x_{n-1} + Cy_1 y_{n-1}}{\alpha}$$

On the other hand, we have

$$\alpha^{n-1} y_n = y_1 \alpha^{n-2} x_{n-1} + x_1 \alpha^{n-2} y_{n-1}$$

$$y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{\alpha}$$

And then,

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = x_n y_{n-1} - y_n x_{n-1}$$
$$= \frac{x_1 x_{n-1} + Cy_1 y_{n-1}}{\alpha} y_{n-1} - \frac{y_1 x_{n-1} + x_1 y_{n-1}}{\alpha} x_{n-1}$$
$$= \frac{-y_1(x_{n-1}^2 - Cy_{n-1}^2)}{\alpha} = \frac{-y_1 \alpha^2}{\alpha} = -y_1 \alpha$$

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -\alpha y_1$$

Since $z_n = x_n + 2 \pm \alpha$, we have
$z_n = \frac{x_1 x_{n-1} + Cy_1 y_{n-1}}{\alpha} + (2 \pm \alpha)$.

Theorem: 3

Let (x_1, y_1) be the basic solution for the Pell equation $x^2 - Cy^2 = \alpha^2$, and then (x_n, y_n, z_n) satisfy the next repeating relationship.

$$x_n = \left(\frac{2}{\alpha}x_1 - 1\right)(x_{n-1} + x_{n-2}) - x_{n-3} \quad (17)$$

$$y_n = \left(\frac{2}{\alpha}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3} \quad (18)$$

$$z_n = \left(\frac{2}{\alpha}x_1 - 1\right)(z_{n-1} + z_{n-2} - 2(2 \pm \alpha)) - (z_{n-3} - 2(2 \pm \alpha)) \quad (19)$$

Proof.

The proof will be provided by submission to n.

We use (13), (14), and (15), we have

$$x_2 = \frac{x_1^2 + Cy_1^2}{\alpha} = \frac{x_1^2 + x_1^2 - \alpha^2}{\alpha}$$

$$x_2 = \frac{2}{\alpha}x_1^2 - \alpha \quad (20)$$

$$y_2 = \frac{x_1 y_1 + x_1 y_1}{\alpha} = \frac{2}{\alpha}x_1 y_1 \quad (21)$$

$$z_2 = x_2 + (2 \pm \alpha) = \frac{2}{\alpha}x_1^2 - \alpha + (2 \pm \alpha) \quad (22)$$

We use (13), (14), (15), (20), (21) and (22), we have

$$x_3 = \frac{x_1 x_2 + Cy_1 y_2}{\alpha} = \frac{x_1\left(\frac{2}{\alpha}x_1^2 - \alpha\right) + Cy_1\left(\frac{2}{\alpha}x_1 y_1\right)}{\alpha}$$

$$x_3 = x_1\left(\frac{4}{\alpha^2}x_1^2 - 3\right) \quad (23)$$

Indian Journal of Advanced Mathematics (IJAM)
ISSN: 2582-8932 (Online), Volume-2 Issue-1, April 2022

$$y_3=\frac{y_1x_2+x_1y_2}{\alpha}=\frac{y_1\left(\frac{2}{\alpha}x_1^2-\alpha\right)+x_1\left(\frac{2}{\alpha}x_1y_1\right)}{\alpha}$$

$$y_3=y_1\left(\frac{4}{\alpha^2}x_1^2-1\right) \quad (24)$$

$$z_3=x_3+(2\pm\alpha)=x_1\left(\frac{4}{\alpha^2}x_1^2-3\right)+(2\pm\alpha) \quad (25)$$

Then with equations (13), (14), (15), (23), (24) and (25), we getx_4 and y_4

$$x_4=\frac{x_1x_3+Cy_1y_3}{\alpha}$$
$$=\frac{x_1\left[x_1\left(\frac{4}{\alpha^2}x_1^2-3\right)\right]+Cy_1\left[y_1\left(\frac{4}{\alpha^2}x_1^2-1\right)\right]}{\alpha}$$

$$x_4=\frac{8}{\alpha^3}x_1^4-\frac{8}{\alpha}x_1^2+\alpha \quad (26)$$

$$y_4=\frac{y_1x_3+x_1y_3}{\alpha}=\frac{x_1y_1\left(\frac{4}{\alpha^2}x_1^2+\frac{4}{\alpha^2}x_1^2-3-1\right)}{\alpha}$$

$$y_4=x_1y_1\left(\frac{8}{\alpha^3}x_1^2-\frac{4}{\alpha}\right) \quad (27)$$

$$z_4=x_4+(2\pm\alpha)=\frac{8}{\alpha^3}x_1^4-\frac{8}{\alpha}x_1^2+\alpha+(2\pm\alpha) \quad (28)$$

Now to replace (20) and (23) in (17), we have it

$$x_4=\left(\frac{2}{\alpha}x_1-1\right)\left(x_1\left(\frac{4}{\alpha^2}x_1^2-3\right)+\frac{2}{\alpha}x_1^2-\alpha\right)-x_1$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\left(\frac{4}{\alpha^2}x_1^3-3x_1+\frac{2}{\alpha}x_1^2-\alpha\right)-x_1$$
$$x_4=\frac{8}{\alpha^3}x_1^4-\frac{8}{\alpha}x_1^2+\alpha$$

And to substitute (21) and (24) in (18), we have

$$y_4=\left(\frac{2}{\alpha}x_1-1\right)(y_3+y_2)-y_1$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\left(y_1\left(\frac{4}{\alpha^2}x_1^2-1\right)+\frac{2}{\alpha}x_1y_1\right)-y_1$$
$$y_4=x_1y_1\left(\frac{8}{\alpha^3}x_1^2-\frac{4}{\alpha}\right)$$

To replace the last (22) and (25) in (19), we have

$$z_4=\left(\frac{2}{\alpha}x_1-1\right)(z_3+z_2-2(2\pm\alpha))-(z_1-2(2\pm\alpha))$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\Big(x_1\left(\frac{4}{\alpha^2}x_1^2-3\right)+(2\pm\alpha)+\frac{2}{\alpha}x_1^2-\alpha+(2\pm\alpha)-2(2\pm\alpha)\Big)-(x_1+(2\pm\alpha)-2(2\pm\alpha))$$
$$=\frac{8}{\alpha^3}x_1^4-\frac{6}{\alpha}x_1^2+\frac{4}{\alpha^2}x_1^3-2x_1-\frac{4}{\alpha}x_1^3-\frac{2}{\alpha}x_1^2+3x_1+\alpha-x_1+(2\pm\alpha)$$

$$z_4=\frac{8}{\alpha^3}x_1^4-\frac{8}{\alpha}x_1^2+\alpha+(2\pm\alpha)$$

Which formulae are the same in (26), (27) and (28).So (17), (18) and (19) hold $n=4$. Now we assume that (17), (18) and (19) hold $n\geq 4$ and we show that it holds $n+1$.

Really by (13), (14) and (15) and by guess we have

$$x_{n+1}=\frac{x_1x_n+Cy_1y_n}{\alpha}$$
$$=\frac{x_1\left[\left(\frac{2}{\alpha}x_1-1\right)(x_{n-1}+x_{n-2})-x_{n-3}\right]}{\alpha}+Cy_1\frac{\left[\left(\frac{2}{\alpha}x_1-1\right)(y_{n-1}+y_{n-2})-y_{n-3}\right]}{\alpha}$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\left[\frac{x_1x_{n-1}+Cy_1y_{n-1}}{\alpha}+\frac{x_1x_{n-2}+Cy_1y_{n-2}}{\alpha}\right]-\left[\frac{x_1x_{n-3}+Cy_1y_{n-3}}{\alpha}\right]$$
$$=\left(\frac{2}{\alpha}x_1-1\right)(x_n+x_{n-1})-x_{n-2}$$

$$y_{n+1}=\frac{y_1x_n+x_1y_n}{\alpha}$$
$$=\frac{y_1\left[\left(\frac{2}{\alpha}x_1-1\right)(x_{n-1}+x_{n-2})-x_{n-3}\right]}{\alpha}+x_1\frac{\left[\left(\frac{2}{\alpha}x_1-1\right)(y_{n-1}+y_{n-2})-y_{n-3}\right]}{\alpha}$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\left[\frac{y_1x_{n-1}+x_1y_{n-1}}{\alpha}+\frac{y_1x_{n-2}+x_1y_{n-2}}{\alpha}\right]-\left[\frac{y_1x_{n-3}+x_1y_{n-3}}{\alpha}\right]$$
$$=\left(\frac{2}{\alpha}x_1-1\right)(y_n+y_{n-1})-y_{n-2}$$

$$z_{n+1}=x_{n+1}+2\pm\alpha$$
$$=\frac{x_1x_n+Cy_1y_n}{\alpha}+(2\pm\alpha)$$
$$=\frac{x_1\left[\left(\frac{2}{\alpha}x_1-1\right)(x_{n-1}+x_{n-2})-x_{n-3}\right]}{\alpha}+Cy_1\frac{\left[\left(\frac{2}{\alpha}x_1-1\right)(y_{n-1}+y_{n-2})-y_{n-3}\right]}{\alpha}+(2\pm\alpha)$$
$$=\left(\frac{2}{\alpha}x_1-1\right)\left[\frac{x_1x_{n-1}+Cy_1y_{n-1}}{\alpha}+\frac{x_1x_{n-2}+Cy_1y_{n-2}}{\alpha}\right]-\left[\frac{x_1x_{n-3}+Cy_1y_{n-3}}{\alpha}\right]+(2\pm\alpha)$$

$$=\left(\frac{2}{\alpha}x_1-1\right)(x_n+x_{n-1})-x_{n-2}+(2\pm\alpha)$$

$$=\left(\frac{2}{\alpha}x_1-1\right)\big(x_n+(2\pm\alpha)+x_{n-1}+(2\pm\alpha)-2(2\pm\alpha)\big)-x_{n-2}-(2\pm\alpha)+2(2\pm\alpha)$$

$$=\left(\frac{2}{\alpha}x_1-1\right)\big(z_n+z_{n-1}-2(2\pm\alpha)\big)-\big(z_{n-2}-2(2\pm\alpha)\big)$$

This completes the proof.

We are now looking at another case $x^2-Cy^2=2^{2t}$ without giving proof of it because it can be proved in the same way as the previous theorems proved.

Theorem: 4

Let (x_1,y_1) be the basic solutions of the Pell equation $x^2-Cy^2=2^{2t}$ and

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix}=\begin{pmatrix} x_1 & Cy_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $n\geq 2$. Then the complete solutions of the transcendental equation $z=x+2\pm\sqrt{x^2-Cy^2}$, $(x^2-Cy^2=2^{2t}$) are (x_n,y_n,z_n), where

$$(x_n,y_n,z_n)=\left(2^{t(1-n)}f_n,2^{t(1-n)}g_n,2^{t(1-n)}f_n+(2\pm\alpha)\right)$$

and (x_n,y_n,z_n) satisfy the next repetition relationship

$$x_n=(2^{1-t}x_1-1)(x_{n-1}+x_{n-2})-x_{n-3}$$

$$y_n=(2^{1-t}x_1-1)(y_{n-1}+y_{n-2})-y_{n-3}$$

$$z_n=(2^{1-t}x_1-1)\big(z_{n-1}+z_{n-2}-2(2\pm\alpha)\big)-\big(z_{n-3}-2(2\pm\alpha)\big)$$

III. CONCLUSION

In this paper, we investigate the the Transcendental equation $\sqrt{2z-4}=\sqrt{x+\sqrt{C}y}\pm\sqrt{x-\sqrt{C}y}$, where $x^2-Cy^2=\alpha^2$ or 2^{2t}. It is interesting to see that the researcher can also proceed for further results in this problem.

REFERENCES

1. S.P. Arya, On the Brahmagupta-Bhaskara Equation. *Math. Ed.* 8(1) (1991), 23-27.
2. I.Niven I., HS Zuckerman and HL Montgomery, An introduction to the Theory of Numbers, Fifth Edition, *John Wiley Sons*, Inc., New York, 1991.
3. A. Tekcan, The Pell Equation $x^2-Dy^2=\pm4$. *Appl. Math. Sci.*, 1(8), (2007), pp. 363-369.
4. A. Tekcan B. Gezer and O. Bizim, On the Integer Solutions of the Pell Equation $x^2-dy^2=2^t$. *Int. J. of Mathematical and Computational Sciences*., Vol: 1, No:1, 2007.
5. Amara Chandoul., The Pell Equation $x^2-Dy^2=\pm k^2$. *Advances in Pure Mathematics*, 2011, 1, 16-22. [CrossRef]
6. P. Thirunavukarasu and S. Sriram, "On Transcendental Equation $\sqrt{2z-4}=\sqrt{x+\sqrt{A}\,y}+\sqrt{x-\sqrt{A}\,y}$", International *Journal of Pure&Engg. Mathematics (IJPEM)*, Nov.2014, Vol 2, No. 1, pp 23-29.

AUTHOR PROFILE

Mr. S.Sriram, received B.Sc., M.Sc., M.Phil and Ph.D., degree in Mathematics received from Bharathidasan University, Tiruchirappalli, Tamilnadu, India.His ongoing research focusing on Number Theory and it's applications in Graphs. . He is working as an Assistant Professor in the Department of Mathematics, National College (Bharathidasan University), Trichy Tamilnadu. He published more than twenty papers in International Journals

Mr. P.Veeramallan, received the B.Sc., degree in Mathematics from Periyar University and M.Sc., M.Phil and Ph.D., degree in Mathematics received from Bharathidasan University, Tiruchirappalli, Tamilnadu, India. He is working as Post Graduate Assistant in Mathematics in GHSS, Yethapur, Salem and doing Ph.D (Part-Time) in National College (Bharathidasan University), Trichy Tamilnadu. His ongoing research is on Number Theory under the supervision of Dr.S.Sriram and publish six papers.