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On Symmetric Riemann-Derivatives

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***Abstract:** The basic properties like monotoni city, Darboux property, mean value property of symmetric Riemann-derivatives of order n of a real valued function f at a point x of its domain (a closed interval) is studied. In some cases, function is considered to be continuous or semi-continuous.*

***Keywords:** Symmetric derivatives, Riemann-derivatives, symmetric Riemann-derivatives. Mathematics Subject Classification (2020): 26E99, 28E99.*

I. INTRODUCTION

Symmetric Riemann-derivatives is a generalization of normal or ordinary or general symmetric derivatives. It is a blender of Riemann-derivatives and symmetric derivatives to form a new generalization of derivatives. Mathematicians worked towards Symmetric derivatives and Riemann-derivatives, then got a new outlook towards symmetric Riemann-derivatives. In 1954, P. L. Butzer and W. Kozakiewicz showed their work on the Riemann-derivatives for integrable functions [7]. Later J. Marshall Ash, Stefan Catoiu and William Chin William worked on generalization of Riemann-derivatives and classification of generalized Riemann-derivatives(1967) [1] [2]. In 1974, P. S. Bullen and S. N. Mukhopadhyay discovered relation between different generalized derivatives [6]. From 1970 and 1974, N. K. Kundu researched on properties of symmetric derivatives including conditions on a function's symmetric derivatives for monotonicity [10] [11]. Around 1972, C. L. Belna , M. J. Evans and P. D. Humke , on symmetric and ordinary differentiation [3]. Sorin Radulescu, Petrus Alexandrescu and Diana-Olimpia Alexandrescu published their paper on generalized Riemann-derivatives and it's reference to study of qualitative property of a function in 2013 [13] [14]. Subhankar Ghosh worked on same field in his Ph. D. Thesis, namely SOME STUDIES ON HIGHER ORDER GENERALIZED DERIVATIVES, SYMMETRIC DERIVATIVES, DIVIDED DIFFERENCES AND THEIR INTERRELATIONS to Visva-Bharati University in 2017 [8]. Many mathematicians such as B. S. Thompson [15], R. G. Bertle and D. R. Sherbert [4], A. Zygmund [16], A. Gordon Russel [9], S. N. Mukhopadhyay [12], A. M. Bruckner [5] compiled the findings so far in books or papers along with something new. In this section we have studied n^{th} order symmetric Riemann-derivatives and have shown by example that symmetric Riemann-derivatives is more general than symmetric derivative.

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Also we have proved some theorems regarding monotonicity and mean value theorem for symmetric Riemann-derivatives of a function having upper semi-continuity and with property D as well as it's relation with Riemann-derivatives.

II. DEFINITIONS AND NOTATIONS

Definition 2.1. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a function. If* $\limsup_{h\to 0} \frac{\Delta_n^s(f,x,h)}{h^n}$ *exists, where* $\Delta_n^s(f,x,h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f\left(x - \frac{nh}{2} + ih\right)$, *then this limit is said to be the n-th upper symmetric Riemann-derivative of f at x and is denoted by* $\overline{SRD}_n f(x)$. *Similarly, the limit* $\liminf_{h\to 0} \frac{\Delta_n^s(f,x,h)}{h^n}$, *if exists, is said to be the n-th lower symmetric Riemann-derivative of f at x and is denoted by* $\underline{SRD}_n f(x)$.

If both $SRD_n^+ f(x)$ *and* $\underline{SRD}_n^- f(x)$ *exist and are equal, then this common value is said to be the n-th symmetric Riemann-derivative of f at x and is denoted by* $SRD_n f(x)$.

Example 2.2. *(i) Let* $f(x) = e^x$.

$$\Delta_n^s(f,x,t) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f\left(x - \frac{nt}{2} + it\right)$$

$$= \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} e^{x - \frac{nt}{2} + it}$$

$$= e^x \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} e^{(i-n)t}$$

$$\lim_{t\to 0} \frac{\Delta_n^s(f,x,t)}{t^n} = e^x \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} \frac{e^{(i-n)t}}{t^n}$$

(ii) Let $f(x) = \sin x$.

$$\Delta_n^s(f,x,t) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f\left(x - \frac{nt}{2} + it\right)$$

$$= \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} \sin\left\{x - \frac{nt}{2} + it\right\}$$

$$=\sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i}[\sin x\cos(i-n)t+\cos x\sin(i-n)t]$$

$$\lim_{t\to 0}\frac{\Delta_n^s(f,x,t)}{t^n} = \sin x\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\left[\lim_{t\to 0+}\frac{\cos(i-n)t}{t^n}\right] + \cos x\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\left[\lim_{t\to 0+}\frac{\sin(i-n)t}{t^n}\right]$$

(iii) Let $f(x) = x^2$.

$$\Delta_1^s(f,x,t) = f(x+\tfrac{t}{2}) - f(x-\tfrac{t}{2}) = (x+\tfrac{t}{2})^2 - (x-\tfrac{t}{2})^2 = 4x\tfrac{t}{2} = 2xt$$

$$\Delta_2^s(f,x,t) = \Delta^s f(x+\tfrac{t}{2}) - \Delta^s f(x-\tfrac{t}{2})$$

$$= f(x+t) - 2f(x) + f(x-t)$$

$$= (x+t)^2 - 2(x)^2 + (x-t)^2$$

$$= 2t^2$$

$$\Delta_n^s(f,x,t) = 0 \text{ if } n > 2.$$

So, $SRD_1 f(x) = 2x = f'(x) = RD_1 f(x)$, $SRD_2 f(x) = 2 = f''(x) = RD_2 f(x)$.

Note 2.3. *Let*

$$f(x) = x^2\sin\tfrac{1}{x} \quad when\ x \in Q$$
$$= x^3 \quad when\ x \in Q$$

Then $f''(0)$, $f_2(0)$, $SDf^2(0)$ *do not exist. But*

$SRD_2 f(0)$

$$= \lim_{t\to 0+}\frac{f(2t)-2f(t)+f(0)}{t^2} = \lim_{t\to 0+}\frac{8t^3-2t^3}{t^2} = \lim_{t\to 0+}\frac{6t^3}{t^2} = 6\lim_{t\to 0+}\frac{t^3}{t^2} = 6\lim_{t\to 0+} t = 0$$

So, symmetric Riemann-derivative is more general than ordinary derivative, Peano derivative, symmetric derivative.

III. SOME RESULTS

Theorem 3.1. *Let* f *be a continuous real valued function in* $[a,b]$, SRD_1^+f *and* SRD_1^-f *exist in a set E contained in* $[a,b]$ *, then* SRD_1^+f , $SRD_1^-f \in B_1(E)$. *Moreover, if (i)* SRD_nf *is finite , (ii)* SRD_if *is continuous in E ,* $i = 0,1,...,n$ *, (iii)* SRD_{n+1}^+f *and* SRD_{n+1}^-f *exist in E , then* SRD_{n+1}^+f, $SRD_{n+1}^-f \in B_1(E)$.

Proof. Let be a function which is continuous in $[a,b]$, SRD_n^+f and SRD_n^-f exist in a set E contained in $[a, b]$.

Since f is continuous in $[a,b]$, SRD_1^+f and SRD_1^-f exist in E, f is differentiable in E.

Suppose $F_n(x) = \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h}$, $h = \frac{1}{n}$. It is obvious that $F_n(x)$ is continuous in E.

$$\lim_{n\to\infty}\lim_{n\to\infty}F_n(x) = \lim_{h\to 0+}\frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h} = SRD_1^+f(x)$$

So, $SRD_1^+f(x) \in B_1(E)$.

Suppose $G_n(x) = \frac{f(x-\frac{h}{2})-f(x+\frac{h}{2})}{h}$, $h = \frac{1}{n}$. It is obvious that $G_n(x)$ is continuous in E.

$$\lim_{n\to\infty} G_n(x) = \lim_{h\to 0-}\frac{f(x-\frac{h}{2})-f(x+\frac{h}{2})}{h} = SRD_1^-f(x)$$

So, $SRD_1^-f(x) \in B_1(E)$.

Suppose, moreover, if (i) $SRD_n f$ is finite, (ii) $SRD_i f$ is continuous in E, $i = 0,1,...,n$, (iii) SRD_{n+1}^+f and SRD_{n+1}^-f exist in E.

Suppose $\Phi_m(x) = \frac{\Delta_{n+1}^s(f,x,h)}{h^{n+1}}$, $h = \frac{1}{m}$. It is obvious that $\Phi_m(x)$ is continuous in E.

$$\lim_{m\to\infty}\Phi_m(x) = \lim_{h\to 0}\frac{\Delta_{n+1}^s(f,x,h)}{h} = SRD_{n+1}^+f(x)$$

So, $SRD_{n+1}^+f(x) \in B_1(E)$.

Suppose $\Psi_m(x) = \frac{\Delta_{n+1}^s(f,x,-h)}{(-h)^{n+1}}$, $h = \frac{1}{m}$. It is obvious that $\Psi_n(x)$ is continuous in E.

$$\lim_{m\to\infty}\Psi_m(x) = \lim_{h\to 0}\frac{\Delta_{n+1}^s(f,x,-h)}{(-h)^{n+1}} = SRD_{n+1}^-f(x)$$

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So, $SRD_{n+1}^{-}f(x) \in B_1(E)$.

Note 3.2. *Let f be a function in [a,b]. If f is non-decreasing in [a,b], then* $SRD_n f(x) \geq 0$ *in* $[a,b]$.

Proof. Suppose $\alpha,\beta \in [a,b]$, such that $\alpha < \beta$. So, $f(\alpha) \leq f(\beta)$.

Now, for any $x_0 \in (a,b)$ and for any δ satisfying $0 < \delta < (b - x_0)$, we have

$f(x_0 - \frac{nh}{2} + \delta) \geq f(x_0 - \frac{nh}{2})$.

$$\Delta_n^s(f, x,h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(x - \tfrac{nh}{2} + ih)$$

Let us take $h\,(> 0)$ in a way such that $max\{0,h,2h,...,(n - 1)h\} \leq \delta$.

Hence,

$$\Delta_n^s(f, x_0,h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(x_0 - \tfrac{nh}{2} + ih)$$

$$\Rightarrow \Delta_n^s(f, x_0,h) \geq \sum_{i=0}^{n-1} (-1)^{n-i}\binom{n}{i} f(x_0 - \tfrac{nh}{2}) + f(x_0 - \tfrac{nh}{2} + nh)$$

$$\Rightarrow \Delta_n^s(f, x_0,h) \geq f(x_0)\sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} + f(x_0 - \tfrac{nh}{2} + nh) - f(x_0 - \tfrac{nh}{2})$$

$$\Rightarrow \Delta_n^s(f, x_0,h) \geq f(x_0 - \tfrac{nh}{2})(-1 + 1)^n + f(x_0 - \tfrac{nh}{2} + nh) - f(x_0 - \tfrac{nh}{2})$$

$$\Rightarrow \Delta_n^s(f, x_0,h) \geq f(x_0 - \tfrac{nh}{2} + nh) - f(x_0 - \tfrac{nh}{2}) \geq 0$$

Then $SRD_n f(x) = \lim_{h\to 0+} \frac{\Delta_n^s(f,x,h)}{h^n} \geq 0$, provided the limit exists.

Theorem 3.3. *Let f be an upper semi-continuous function which has the property D in [a,b]. If* $E = \{x \in [a,b] : SRD_n^{+} f(x) f \leq 0\}$ *and f (E) has no sub-interval, then f is non-decreasing in [a,b].*

Proof. Suppose $\alpha, \beta \in [a,b]$, such that $\alpha < \beta$. So, $f(\alpha) > f(\beta)$.

Now, let $y_0 \in (f(\alpha), f(\beta))$ such that y_0 doesn't belong to $f(E)$.

Let $S = \{x \in [a,b] : f(x) \geq y_0\}$ and $x_0 =$ sup S.

Since f is an upper semi-continuous function with property D in $[a,b]$, S is closed and thus $x_0 \in S$. Therefore, $f(x_0) \geq y_0$. We will show that $f(x_0) = y_0$.

If not, there exist η satisfying $f(\beta) < y_0 < \eta < f(x_0)$ and $\xi \in (x_0, \beta)$, such that $f(\xi) = \eta$. It contradicts that $x_0 = sup\ S$.

So, $f(x_0) = y_0$.

Since f is an upper semi-continuous function with property D in $[a,b]$ and $x_0 < \beta$,

for $x_0 < x < \beta$, $f(x) < f(x_0)$.

If $0 < \delta < (\beta - x_0)$, then $f(x_0 + \delta) - f(x_0) < 0$.

Again, f being upper semi-continuous function with property D in $[a,b]$, for any $y_0 > y$ there is a neighbourhood U of x_0 such that $y < f(x) < y_0$, whenever $x \in U$.

$$\Delta_n^s(f, x_0, h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(x_0 - \tfrac{nh}{2} + ih)$$

Let us take $h\,(> 0)$ in a way such that

$x_0 - \frac{nh}{2} + ih \in Ux_{0+}\frac{nh}{2}$ for all $i = 0,1,...,n$ and

$max\{0, h,2h,...,(n - 1) h\} \leq \delta$.

Therefore,

$$\Delta_n^s(f, x_0,h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(x_0 - \tfrac{nh}{2} + ih)$$

$$\Rightarrow \Delta_n^s(f,x_0,h) < \sum_{i=0}^{n-1} (-1)^{n-i}\binom{n}{i} f(x_0 - \tfrac{nh}{2}) + f(x_0 + \tfrac{nh}{2})$$

$$\Rightarrow \Delta_n^s(f, x_0, h) < f(x_0 \tfrac{nh}{2})\sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} + f(x_0 + \tfrac{nh}{2}) - f(x_0 - \tfrac{nh}{2})$$

$$\Rightarrow \Delta_n^s(f, x_0, h) < f(x_0)(-1 + 1)^n + f(x_0 + \tfrac{nh}{2}\ nh) - f(x_0 - \tfrac{nh}{2})$$

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$\Rightarrow \Delta_n^{s}(f, x_0, h) < f(x_0 + \frac{nh}{2}) - f(x_0 - \frac{nh}{2}) < 0$

Then $SRD_n f(x_0) = \lim_{h\to0+} \frac{\Delta_n^s(f,x_0,h)}{h^n} \le 0$, implies $x_0 \in S$ and hence $y_0 \in E$, a contradiction. So, our initial assumption is wrong. There can not be α, $\beta \in [a,b]$, such that $\alpha < \beta$. So, $f(\alpha) > f(\beta)$. So, f is non-decreasing in $[a,b]$.

Theorem 3.4. *Let f be an upper semi-continuous function which has the property D in $[a,b]$, $SRD_n f(x) \ge 0$ in $[a,b]$ except an enumerable set E. Then f is non-decreasing in $[a,b]$.*

Proof. Suppose $\epsilon > 0$ be arbitrarily small number and $g(x) = f(x) + \epsilon x$.

$SRD_n^+ g(x) = \lim_{h\to0+} \frac{\Delta_n^s(g,x,h)}{h^n} = \lim_{h\to0+} \frac{\Delta_n^s(f,x,h)}{h^n} + \epsilon \lim_{h\to0+} \frac{\Delta_n^s(I,x,h)}{h^n}$, where $I(x) = x$

$$\Rightarrow SRD_n^+ g(x) \le SRD_n^+ f(x) + \epsilon$$

as $\Delta_n^s(I, x, h) = 1$ if $n = 1$ and $\Delta_n^s(I,x,h) = 0$ if $n \ge 2$,

$$\Rightarrow SRD_n^+ g(x) = SRD_n^+ f(x)$$

Here, g is also an upper semi-continuous function with property D in $[a,b]$, moreover $g(E)$ is measurable thus contains no sub-interval. So, g is non-decreasing in $[a,b]$. Since ϵ is arbitrarily small positive number, f is non-decreasing in $[a,b]$.

Theorem 3.5. *Let f be an upper semi-continuous function which has the property D in $[a,b]$, $SRD_n f(x) \ge 0$ almost everywhere in $[a,b]$, $SRD_n^+ f(x) > -\infty$ in $[a,b]$ except an enumerable set E. Then f is non-decreasing in $[a,b]$.*

Proof. Let

$A = \{x \in [a,b] : SRD_n^+ f(x) < 0\}$. Clearly, $m(A) = 0$.

Suppose σ is a continuous, non-decreasing function in $[a,b]$ such that $\Delta_n^s(\sigma, x, h) \ge 0$ in $[a,b]$ except A. We consider an arbitrary small positive number ϵ and take $g(x) = f(x) + \epsilon\, \sigma(x)$. Then g an upper semi-continuous function with property D in $[a,b]$,

$SRD_n^+ g(x)$

$= \lim_{h\to0+} \frac{\Delta_n^s(g,x,h)}{h^n}$

$= \lim_{h\to0+} \frac{\Delta_n^s(f,x,h)}{h^n} + \epsilon \lim_{h\to0+} \frac{\Delta_n^s(\sigma,x,h)}{h^n}$

$= SRD_n^+ f(x) + \epsilon\, SRD_n^+ \sigma(x)$,

Therefore, $SRD_n^+ g(x) \ge 0$ almost everywhere in $[a,b]$ except A. Hence, g is non-decreasing in $[a,b]$. Since ϵ is arbitrarily small positive number, f is non-decreasing in $[a,b]$.

Note 3.6. *Example of a function σ which is continuous, non-decreasing in $[a,b]$ such that $\Delta^n(\sigma, x, h) \ge 0$ in $[a,b]$ except a set A of measure zero is a polynomial $ax^k + bx^{k-2} + \dots + \lambda$, where the co-efficients are all positive and k is an even natural number.*

Theorem 3.7. *If f is continuous and $SRD_n f(x)$ exists in $[a,b]$ then $SRD_n^+ f(x)$ has Darboux property in $[a,b]$.*

Proof. Let us consider that $SRD_n^+ f(x)$ does not have Darboux property, then there exist α, β such that $f(\alpha) < 0$, $f(\beta) > 0$ but $SRD_n^+ f(x) \neq 0$ for any $x \in (\alpha,\beta)$. Further, suppose $E^+ = \{x \in [\alpha,\beta] : SRD_n^+ f(x) > 0\}$, $E^- = \{x \in [\alpha,\beta] : SRD_n^+ f(x) < 0\}$, then

$[\alpha, \beta] = E^+ \cup E^-$.

Let Q be (if any) non-degenerate component of E^+. Then Q is an interval. Suppose c,d be the end points of Q. $SRD_n^+ f > 0$ in Q, so f is non-decreasing in Q. f being continuous and non-decreasing in $[c,d]$, $SRD_n^+ f(c), SRD_n^+ f(d) > 0$. Therefore $c,d \in Q$, implies that Q is a closed interval. Q being arbitrary, every non-degenerate component of E^+ is a closed interval. Following similar arguments, it can be shown that every non-degenerate component of E^- is a closed interval. Let Q^+, Q^- be the collection of all non-degenerate components of E^+ and E^- respectively.Let $\mathbf{Q} = Q^+ \cup Q^-$. Then any two distinct members of Q are disjoint. Hence, $P = [\alpha,\beta] - \cup Q^0$, $Q \in \mathbf{Q}$, is perfect and $SRD_n^+ f$ has no point of continuity in P relative to P, which is a contradiction as $SRD_n^+ f \in B[\alpha,\beta]$. Therefore, $SRD_n^+ f(x)$ must have Darboux property.

Theorem 3.8. *If f is continuous in $[a,b]$ and $SRD_n f(x)$ exists in (a,b) then there exists $c \in (a,b)$ such that $f(b) - f(a) = (b - a)\, SRD_n f(c)$.*

Proof. Here we may have following two cases -

Case 1:

Let $f(b) = f(a)$. Then,

Subcase 1- In case $SRD_n f(x) \ge 0$ or $SRD_n f(x) \le 0$ in (a,b). Thus f is monotone function. Now f being continuous as well as monotone, f is constant in (a,b), ensuring the existence of c.

Subcase 2-In case f is not monotone, $SRD_n f(\alpha) < 0$ and $SRD_n f(\beta) > 0$ for some $\alpha,\beta in(a,b)$ and hence there exists $\xi \in (a,b)$ such that $SRD_n f(\xi) = 0$, implying $c = \xi$.

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Case 2:

Let $f(b) \neq f(a)$. Then, suppose $\Phi(x) = f(x) - Ax$, A=constant. Clearly, Φ is continuous in $[a,b]$ and $SRD_n \Phi(x)$ exists in (a,b).

Also, $SRD_n\Phi(x) = SRD_n f(x)$. Let us take $A = \frac{f(b)-f(a)}{b-a}$. Thus, $\Phi(b) = \Phi(a)$. By Case 1, there exists $c \in (a,b)$ such that $SRD_n \Phi(c) = 0$ $\Rightarrow SRD_n f(c) = \frac{f(b)-f(a)}{b-a}$.

This completes the proof of the theorem.

Note 3.9. *Above results are applicable for any continuous function f.*

IV. CONCLUSION

From above analysis and discussion, it is clear that the symmetric Riemann-derivatives can be a new type of generalized derivatives, can follow many monotonicity, mean value property, Darboux property etc, like ordinary and some other derivatives but only if some conditions are satisfied. We have to work more to decrease the number of these conditions and to find more results on a more generalized derivatives.

Application

The above work on symmetric Riemann-derivatives provides scope of finding new derivatives of a function which are more generalized than ordinary derivatives, even some other generalized derivatives, under less number of restrictions. This work can provide clue for farther findings on Riemann fractional derivatives and can be used in differential equations, specially in electrical and mechanical phenomena analysis.

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DECLARATION

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