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Adam-Bash forth-Multan Predictor Corrector Method for Solving First Order Initial value problem and Its Error Analysis

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***Abstract:** This paper presents fourth order Adams predictor corrector numerical scheme for solving initial value problem. First, the solution domain is discretized. Then the derivatives in the given initial value problem are replaced by finite difference approximations and the numerical scheme that provides algebraic systems of difference equations is developed. The starting points are obtained by using fourth order Runge-Kutta method and then applying the present method to finding the solution of Initial value problem. To validate the applicability of the method, two model examples are solved for different values of mesh size. The stability and convergence of the present method have been investigated. The numerical results are presented by tables and graphs. The present method helps us to get good results of the solution for small value of mesh size h. The proposed method approximates the exact solution very well. Moreover, the present method improves the findings of some existing numerical methods reported in the literature.*

***Keywords:** Initial Value Problem, Adams-Bashfor Method, Adams-Molten Method, Adamsbashfor-Molten, Stability, Convergence.*

I. INTRODUCTION

Differential equations are commonly used for mathematical modeling in science and engineering. Many problems of mathematical physics can be started in the form of differential equations. These equations also occur as reformulations of other mathematical problems such as ordinary differential equations and partial differential equations [1]. It is a known fact that several mathematical models emanating from the real and physical life situations cannot be solved explicitly, one has to compromise at numerical approximate solutions of the models achievable by various numerical techniques of different characteristics [19].

Numerical methods are generally used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations. Even then there exist a large number of ordinary differential equations whose solutions cannot be obtained in closed form by using well-known analytical methods, where we have to develop and use the numerical methods to get the approximate solution of a differential equation under the prescribed initial condition. Development of numerical methods for the solution of initial value problems in ordinary differential equations has attracted the attention of many researchers in recent years. Many authors have derived new numerical integration methods, giving better results than a few of the available ones in the literature such as: [7, 11, 12, 17, 19], just to mention a few. From the literature review we may realize that several works in numerical solutions of initial value problems using single step methods have been carried out. Many authors have attempted to solve initial value problems (to obtain high accuracy rapidly by using numerous single-steps methods, such as Euler method and Runge-Kutta method, and also some other methods. In [2] the author discussed accuracy analysis of numerical solutions of initial value problems for ordinary differential equations, and also in [3] the author discussed accurate solutions of initial value problems for ordinary differential equations with fourth-order Range-Kutta method. In [4] the other studied on some numerical methods for solving initial value problems in ordinary differential equations. Moreover in [5 - 21] there are different numerical methods applied by the author to solve initial value problems for ordinary differential equations. However, still, the accuracy and stability of the method need attention because of the treatment of the method used to solve the initial value problem is not trivial distribution. Even though the accuracy and stability of the aforementioned methods need attention, they require large memory and long computational time. So the treatments of this method present severe difficulties that have to be addressed to ensure the accuracy and stability of the solution. To this end, the aim of this paper is to develop the accurate and stable fourth order Adams-predictor corrector method that is capable of producing a solution of initial value problem and approximate the exact solution. The convergence has been shown in the sense of maximum absolute error (e_r) and so that the local behavior of the solution is captured exactly. The stability and convergences of the present methods is also investigated.

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The present paper is organized as follows. Section two Description of method, Section three describes formulation of the numerical scheme, Section four is about Stability and convergence analysis, Section five is about numerical examples and results , Section six describe discussion and conclusion and section seven is declaration.

II. DESCRIPTION OF METHODS

In this section we Adams methods for finding the approximate solutions of the initial value problem (IVP) of the first-order ordinary differential equation has the form

$$y' = f(x,y), \quad x \in (x_0,x_n) \ ,y(x_0) = y_0 \qquad (1)$$

Where $y' = \frac{dy}{dx}$, $f(x,y)$ is given smooth faction and $y(x)$ is the solution of the Equation (1). In this paper we determine the solution of this equation on a finite interval (x_0,x_n), starting with the initial point x_0 . A continuous approximation to the solution $y(x)$ will not be obtained; instead, approximations to y will be generated at various values, called mesh points, in the interval (x_0,x_n). Numerical methods employ the Equation (1) to obtain approximations to the values of the solution corresponding to various selected values of $x_n = x_0 + nh$, $n = 1,2,3,\ldots,N-1$ .The parameter h is called the step size. The numerical solutions of (1) is given by a set of points $\{(x_n,y(x_n)), n = 0,1,2,\ldots\ldots\}$ and each point (x_n,y_n) is an approximation to the corresponding point $(x_n,y(x_n))$ on the solution curve.

III. FORMULATION OF NUMERICAL SCHEME

Runge-Kutta Methods

In multistep methods the solution estimate for y_n at x_n can be attained by ultimation the information on two or more previous points rather than one. To solve initial value problem in Eq.(1) by using multistep methods at x_1, the information on at least two previous points are needed. However, the only available information is y_0 . So the multistep method cannot self starting methods. Thus first we apply a classical fourth order Runge-Kutta method to find the starting point. As mentioned in [1] ,the classical fourth order Runge-Kutta method is:

$$y_{n+1} = y_n + {}^1/_6\,(k_1 + 2k_2 + 3k_3 + k_4)$$

$$k_1 = hf(x,y),$$

$$k_2 = hf(x + \tfrac{h}{2}, y + \tfrac{k_1}{2}$$

$$k_3 = hf(x + \tfrac{h}{2}, y + \tfrac{k_2}{2},$$

$$k_2 = hf(x + h, y + k_3) \qquad (2)$$

where n=1,2,3,… N, and N is number of grid point.

Adams Bashfors-Multen Predictor Corrector Methods

This methods are quite accurate, stable and easy to programming. It contain two methods Adams-Bashfor and Adams-Molten method and respectively explicitly and implicitly method. The following algorithm is based on Adams-Bash forth four-step method as a predictor and also an iteration of Adams-Moulton four-step method as a corrector. As it mentioned in [18], this predictor corrector method is :

Predictor:

$$y_{n+1} = y_n + h/24(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

Corrector:

$$y_{n+1}^{(k)} = y_n^{(k)} + h/24\left(9f_{n+1}^{(k)} + 19f_n^{(k)} - 5f_{n-1}^{(k)} + f_{n-2}^{(k)}\right) \qquad (3)$$

where $k = 0,1,2,\ldots.,$ and both local truncation errors of formula in Eq.(3) are:

$$T_n = \frac{251}{720}h^5y^{(5)}(\zeta)$$

$$,T_n^{(k)} = \frac{251}{720}h^5y^{(5)}(\zeta)$$

$$x_0 \le \zeta \le x_N \qquad (4)$$

Now we use the above formula in Eq.(3) to find numerical solution of IVP in (1).

IV. STABILITY AND CONVERGENT ANALYSIS

This journal A numerical solution is said to be if the effect of any single fixed round-off error is bounded, independent of mesh point [21].More precisely if for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon)$, and the difference between two different numerical solutions y_n *and* $\overline{y_n}$ is less than ϵ .It means that:

$$|\,y_n - \overline{y_n}| < \epsilon\,,\ |\,y_0 - \overline{y_0}| < \delta\ ,0 < h < h_0$$

Definition [21]: Let $y(x) = ce^{\lambda x}$ be the exact solution of IVP in Eq.(1) where c is constant .Using initial condition $y(x_0) = y_0$ we can rewriting the exact solution $y(x) = y(x_0)e^{\lambda(x-x_0)} = y(x_0)e^{\lambda h}$. Now the relation of $y(x_n)$ at x_n and x_{n+1} is $(x_{n+1}) = y(x_n)e^{\lambda(x_n)}$, n=1, 2, 3, …,N.

For sufficient value of $|\,\lambda h|$ by Taylor series, the polynomial approximation to $e^{\lambda x}$ with p order is

$$e^{\lambda h} = 1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{3!}(\lambda h)^3 + \frac{1}{4!}(\lambda h)^4 + \cdots + \frac{1}{p!}(\lambda h)^p + O(\lambda h)^{p+1} \qquad (6)$$

The pade approximation of Eq.(6) is

$$e^{\lambda h} = \frac{1+\frac{1}{2}\lambda h+\frac{1}{10}(\lambda h)^2+\frac{1}{120}(\lambda h)^3+\cdots}{1-\frac{1}{2}\lambda h+\frac{1}{10}(\lambda h)^2-\frac{1}{120}(\lambda h)^3+\cdots} + (\lambda h)^p, p = 0,1,2,3,\ldots \qquad (7)$$

Let $E(\lambda h) \approx e^{\lambda h}$ .Thus the numerical approximation of IVP in Eq.(1) is. $y_{n+1} = E(\lambda h)y_n$,

$n = 1,2,3, \dots N-1$ .Now we wants to show that $|E(\lambda h)| \leq 1$.

To show this we show that the roots of recurrence relation of error term are inside or on the unit circle. So conceder that the exact approximation of IVP in Eq. (1) is:

$$y_{n+1} = y_n + h\lambda/24(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) + T_n$$

$$y_{n+1}^{(k)} = y_n^{(k)} + h\lambda/24\left(9f_{n+1}^{(k)} + 19f_n^{(k)} - 5f_{n-1}^{(k)} + f_{n-2}^{(k)}\right) + T_n^{(k)} \qquad (8)$$

where T_n and $T_n^{(k)}$ are local truncation errors in predictor and corrector formula. Subtracts Eq.(3) from Eq.(8) we obtain:

$$\varepsilon_{n+1} = \varepsilon_n + \Psi/24(55\varepsilon_n - 59\varepsilon_{n-1} + 37\varepsilon_{n-2} - 9\epsilon_{n-3}) + T_n$$

$$\varepsilon_{n+1}^{(k)} = \varepsilon_n^{(k)} + \Psi/24\left(9\varepsilon_{n+1}^{(k)} + 19\varepsilon_n^{(k)} - 5\varepsilon_{n-1}^{(k)} + \varepsilon_{n-2}^{(k)}\right) + T_n^{(k)} \qquad (9)$$

where $k = 0,1,2, \dots,$ ε_{n+1} and $\varepsilon_{n+1}^{(k)}$ are respectively error terms of predicted and corrected numerical solution and $\Psi = \lambda h$. The Eq. (9) are an inhomogeneous forms of difference equation error with the constant coefficients . The general solution of Eq.(9) will consists of particular solution terms. To find this solution, take $T = (T_n, T_n^{(k)}) = 0$,then the homogeneous form of difference equation of error terms is:

$$\varepsilon_{n+1} = \varepsilon_n + \Psi/24(55\varepsilon_n - 59\varepsilon_{n-1} + 37\varepsilon_{n-2} - 9\epsilon_{n-3})$$

$$\varepsilon_{n+1}^{(k)} = \varepsilon_n^{(k)} + \Psi/24\left(9\varepsilon_{n+1}^{(k)} + 19\varepsilon_n^{(k)} - 5\varepsilon_{n-1}^{(k)} + \varepsilon_{n-2}^{(k)}\right) \qquad (10)$$

Looking the error solution of Eq.(10) in form of:

$$\varepsilon_n = A\xi_n, \quad \varepsilon_n^{(k)} = B\xi_n^{(k)} \qquad (11)$$

where $A \neq 0$ and $B \neq 0$ are constant and $\xi = (\xi_n\ \xi_n^{(k)})$ are constant to be determined. Substituting Eq.(11) in to Eq.(10) , respectively we obtain:

$$A\xi_{n+1} = A\xi_n + \Psi/24(55A\xi_n - 59A\xi_{n-1} + 37A\xi_{n-2} - 9A\xi_{n-3})$$

$$B\xi_{n+1}^{(k)} = B\xi_n^{(k)} + \Psi/24\left(9B\xi_{n+1}^{(k)} + 19B\xi_n^{(k)} - 5B\xi_{n-1}^{(k)} + B\xi_{n-2}^{(k)}\right) \qquad (12)$$

Simplifying Eq.(12) we obtain

$$\xi_{n+1} - \left(1 + \frac{55\Psi}{24}\right)\xi_n + \frac{59\Psi}{24}\xi_{n-1} - \frac{37\Psi}{24}\xi_{n-2} + \frac{9\Psi}{24}\xi_{n-2}\xi_{n-3} = 0$$

$$\xi_{n+1}^{(k)} - \left(\frac{24+19\Psi}{24-9\Psi}\right)\xi_n^{(k)} + \frac{5\Psi}{24-9\Psi}\xi_{n-1}^{(k)} - \frac{\Psi}{24-9\Psi}\xi_{n-2}^{(k)} = 0 \qquad (13)$$

Let us conceder that ξ_1, ξ_2, ξ_3 and ξ_4 are distinct four roots of characteristics equation of error term in Eq.(13), then the solution of Eq.(10) is written as: $c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4\xi_4$ and $a_1\xi_1^{(k)} + a_2\xi_2^{(k)} + a_3\xi_3^{(k)} + a_4\xi_4^{(k)}$ respectively where $a = (a_1, a_2, a_3\ , a_4)$ and $c=(c_1, c_2\ , c_3\ , c_4)$ are arbitrary constant to be determined from initial error. If the characteristic equation has double root , $\xi_1 = \xi_2$, the solution of Eq.(10) is $(c_1 + nc_2)\xi_1 + c_3\xi_3 + c_4\xi_4$ and $(a_1 + na_2)\xi_1^{(k)} + a_3\xi_3^{(k)} + a_4\xi_4^{(k)}$. if all roots are equal, the solution of Eq.(10) are $(c_1 + nc_2 + n^2c_3 + n^3c_4)\ \xi_1$ and $(a_1 + na_2 + n^2a_3 + n^3a_4)\ )\xi_1^{(k)}$.

Now to find particular solution for inhomogeneous parts, assume that $T = T_n$ and $T_1^{(k)} = T_n^{(k)}$ constant, the particular solution of difference equation of error terms in Eq.(10) is $\frac{T}{\Psi}$. Therefore the general solution of of Eq.(10) is:

$$\varepsilon_{n+1} = c_1\xi_{1n} + c_2\xi_{2n} + c_3\xi_{3n} + c_4\xi_{4n} + \frac{T}{\Psi}$$

$$\varepsilon_{n+1}^{(k)} = a_1\xi_{1n}^{(k)} + a_2\xi_{2n}^{(k)} + a_3\xi_{3n}^{(k)} + a_4\xi_{4n}^{(k)} + \frac{T_1^{(k)}}{\Psi} \qquad (14)$$

Thus from Eq.(14) to show the stability of method we have $|\ \varepsilon_n| < \infty$ and $\left|\varepsilon_n^{(k)}\right| < \infty$ for $n \to \infty$. Then the error terms is bounded. So that the method is stable.

Definition [18]: A multistep method when applying $y' = \lambda y$, $\lambda < 0$ is said to be absolutely stable if the root of characteristics equation of homogeneous difference equation are lies either inside unit circle or on the unit circle and simple.

Definition[18]: A numerical approximation of IVP in Eq.(1) of the form $y_{n+1} = E(\lambda h)y_n$ is said to be convergent, If $\lim_{h\to 0} y_n = y(x_n)$ for $x_0 \leq x_n \leq x_N$, n=0,1,2,3,…,N-1.The true value $y(x_n)$ satisfy

$$y(x_{n+1}) = E(\lambda h)y(x_n) + T_n \qquad (15)$$

where T is local truncation error. Thus the approximation solution is also satisfy

$$y_{n+1} = E(\lambda h)y_n - R_n\ , y_{n+1} = E(\lambda h)y_n - R_n^{(k)} \qquad (16)$$

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where R is rounding error. Subtracting Eq.(16) from Eq.(15) we obtain

$$\varepsilon_{n+1} = E(\lambda h)\varepsilon_n + R_n + T_n \quad ,$$
$$\varepsilon_{n+1}^{(k)} = E(\lambda h)\varepsilon_n^{(k)} + T_n^{(k)} + R_n^{(k)} \qquad (17)$$

Assume that $\max_n |R_n| = R$ and $\max_n |T_n| = T$ are constant and from Eq.(17) by triangular in equality, we obtain:

$$|\varepsilon_{n+1}| \leq |E(\lambda h)\varepsilon_n| + R_n + T_n \, ,$$
$$|\varepsilon_{n+1}^{(k)}| \leq |E(\lambda h)\varepsilon_n^{(k)}| + T_n^{(k)} + R_n^{(k)}$$
$$, k = 0,1,2 \; .... \qquad (18)$$

By induction Eq.(18) become for $E(\lambda h) \neq 0$ which is constant

$$|\varepsilon_n| \leq E^n(|\varepsilon_0| + \left(\frac{E^n(\lambda h) - 1}{E(\lambda h) - 1}\right)(R_n + T_n) \, ,$$

$$|\varepsilon_n^{(k)}| \leq E^n(\lambda h)|\varepsilon_0^{(k)}| + \left(\frac{E^n(\lambda h) - 1}{E(\lambda h)}\right)\left(T_n^{(k)} + R_n^{(k)}\right) \qquad (19)$$

Let $E(\lambda h)$ be the p^{th} order approximation, then $e^{\lambda h} = E(\lambda h) + \frac{(\lambda h)^{p+1}}{(p+1)!} M_{p+1}$ where $M_{p+1} = \max_{x_n \leq \xi \leq x_n} |y^{(P+1)}(\xi)|$ is a constants. Thus the local truncation error is $T_{p+1} \leq \frac{251}{720} M_{p+1}$. Therefore Eq.(19) becomes:

$$|\varepsilon_n| \leq e^{\lambda(x_n - x_0)}|\varepsilon_0| + \left(\frac{e^{\lambda(x_n - x_0)} - 1}{\lambda\left(1 + \frac{\lambda h}{2!} + \frac{(\lambda h)^2}{3!} + \cdots + \frac{(\lambda h)^{p-1}}{p!}\right)}\right)\left(\frac{R_n}{h} + \frac{(\lambda h)^p}{(P+1)!} M_{p+1}\right)$$

$$|\varepsilon_n^{(k)}| \leq e^{\lambda(x_n - x_0)}|\varepsilon_0^{(k)}| + \left(\frac{e^{\lambda(x_n - x_0)} - 1}{\lambda\left(1 + \frac{\lambda h}{2!} + \frac{(\lambda h)^2}{3!} + \cdots + \frac{(\lambda h)^{p-1}}{p!}\right)}\right)\left(\frac{R_n^{(k)}}{h} + \frac{(\lambda h)^p}{(P+1)!} M_{p+1}\right) \qquad (20)$$

$p = 0,1,2,3, ...$

In both case of Eq.(20) we can seen that for $h \to 0$, the truncation error terms are bounded . Which means that the error terms $\varepsilon_n < \varepsilon_0$ and $\varepsilon_n^{(k)} < \varepsilon_0^{(k)}$ for $= 0,1,2,3, \ldots and \; n = 1,2,3, \ldots., N$. If $|\varepsilon_0| = 0$ and $p = 1$,Eq.(20) become

$$|\varepsilon_n| \leq \left(\frac{e^{\lambda(x_n - x_0)} - 1}{\lambda}\right)\left(\frac{R_n}{h} + \frac{\lambda h}{2} M_2\right) \, ,$$
$$|\varepsilon_n^{(k)}| \leq \left(\frac{e^{\lambda(x_n - x_0)} - 1}{\lambda}\right)\left(\frac{R_n^{(k)}}{h} + \frac{\lambda h}{2} M_2\right)$$

Hence if $h \to 0$ the truncation error is tend to zero where as round-off error become infinity. So to avoided this condition, let us choice the value of h such that $h \approx \sqrt{\frac{2R}{\lambda M_2}}$. Thus the round-off error in this equation is bounded. So the general error terms are bounded and Then this show that the method is stable by choosing appropriate value for space size h.

To show the convergent of our method in Eq.(3), we must determine the constant c . Now let us consider that general error terms $T_j = \varepsilon - \frac{T}{h}$, $j = 0,1,2,3$,the constant can be found by solving the linear system of equation :
$E_1 = c_1 + c_2 + c_3 + c_4$, $E_2 = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4\xi_4$, $E_3 = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4\xi_4$ and $E_4 = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4\xi_4$

Assuming that the initial errors are ε_0, ε_1, ε_2, and ε_3 are constants and equal to ε, by using Lagrange interpolation formula from Eq. (9) we have

$$\varepsilon_n = \left(\varepsilon - \frac{T}{h}\right)\Bigg[\left(\frac{(1-\xi_2)(1-\xi_3)(1-\xi_4)}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_1-\xi_4)}\right)\xi_1^n + \left(\frac{(1-\xi_1)(1-\xi_3)(1-\xi_4)}{(\xi_2-\xi_1)(\xi_2-\xi_3)(\xi_2-\xi_4)}\right)\xi_2^n + \left(\frac{(1-\xi_2)(1-\xi_1)(1-\xi_4)}{(\xi_3-\xi_2)(\xi_3-\xi_1)(\xi_3-\xi_4)}\right)\xi_3^n + \left(\frac{(1-\xi_2)(1-\xi_3)(1-\xi_1)}{(\xi_4-\xi_2)(\xi_4-\xi_3)(\xi_4-\xi_1)}\right)\xi_4^n\Bigg] + \frac{T}{\Psi} \qquad (21)$$

Now from Eq.(21) as $h \to 0$ and $\xi_1 \to 1$, ξ_2 , ξ_3 and ξ_4 are approach to zero. For sufficiently small value of $|\lambda h|$, $\xi_1 \approx e^{\lambda h}$, and all ξ_2 , ξ_3 and ξ_4 are less than one. Thus Eq.(21) become $\varepsilon_n = \varepsilon e^{\lambda h} + \frac{T}{\Psi}\left[1 - e^{\lambda h}\right]$. This implies that $\varepsilon_n \leq \frac{251}{720} h^5 M_5 \left[1 - e^{\lambda h}\right]$. For $M_5 = \max_{x_0 \leq \xi \leq x_n} |y^{(5)}(\xi)|$ .Hence putting $|\varepsilon_0| = 0$, our local truncation error is $|T| \leq \frac{251}{720} h^5 M_5$. Therefore this shows that $|\varepsilon_n| \to 0$ as $h \to 0$ and the method is convergent.

Criteria for Investigating the Accuracy of the Method

In this section, we investigate the accuracy of the present method. There are two types of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors occur when ordinary differential equations are solved numerically. Rounding errors originate from the fact that computers can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers or cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is call Round-off error. Truncation errors in numerical analysis arise when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we make the step size, *h*. To show the accuracy of the present method, maximum absolute error e_r is used to measure the accuracy of the method. The maximum absolute error are calculated as followed in [1] is given by $e_r = \max_{1 \leq n \leq N}(|y(x_n) - y_n|)$

where N is maximum number of step, $y(x_n)$ is exact solution and y_n approximation solution of IVP in Eq.(1) at grid point x_n.

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V. NUMERICAL EXPERIMENTS

In order to test the validity of the proposed method, we have considered the following two model problem. Numerical results and errors are computed and the outcomes are represented by tabular and graphically.

Example 1: we consider the initial value problem considered by [1] $y'(x) = x^2 + .$, $y(0) =$ on the interval $0 \le x \le .$ The exact solution of the given problem is given by $y(x) = \sqrt{\frac{\pi}{2}} e^{\frac{}{2}} erf\left(\frac{x}{2}\right) + e^{\frac{}{2}} -$

Example 2: we consider the initial value problem considered by [1] $y'(x) = xy - ,$ $y(0) =$ on the interval $0 \le x \le .$ The exact solution of the given problem is given by $y(x) = \frac{}{}$
research of the title.

Table 1.Maximum Absolute error , exact $y(x$ and approximation solution for Example 1 with a uniform mesh size $h = ($

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Multan methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	1.00000	5.3465E-03	1.00534648	4.16045E-08	1.005346462	3.782227E-10	1.00534652
0.2	1.011	1.1889E-02	1.0228893	8.26716E-08	1.022889379	7.5155961E-10	1.0228894624
0.3	1.035219	1.9972E-02	1.0551918	1.23029E-07	1.055191840	1.118444E-09	1.05519196
0.4	1.0752765	3.0042E-02	1.10531878	1.63325E-07	1.1053219852	2.756594E-08	1.1053189529
0.5	1.13428766	4.2687E-02	1.176974	2.05805E-07	1.176981986	6.376484E-08	1.1769749725
0.6	1.21600204	5.86769E-02	1.27467873	2.55624E-07	1.2746910864	1.0994988E-07	1.2746789919
0.7	1.3249621	7.90261E-02	1.40398799	3.23030E-07	1.404006972	1.695829E-07	1.4039883184
0.8	1.4667095	1.05078E-01	1.57178734	4.26941E-07	1.5718149601	2.4718573E-07	1.5717877696
0.9	1.64804628	1.38620E-01	1.78666525	6.00769E-07	1.7867042183	3.4827700E-07	1.7866658536
1.0	1.87737044	1.82037E-01	2.0594065	9.01815E-07	2.0594604649	4.8236012E-07	2.059407405

Table 2.Maximum Absolute error , exact $y(x$ and approximation solution for Example 1 with a uniform mesh size $h = 0.$

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Multan methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	1.002625	2.72152E-03	1.00534651	2.59745E-09	1.0053465192	1.2368E-11	1.0053465244
0.2	1.01682416	6.06530E-03	1.02288945	5.15419E-09	1.022889541	3.7629E-10	1.02288946218
0.3	1.04497980	1.02122E-02	1.05519195	7.64925E-09	1.055192238	1.3104E-09	1.05519196376
0.4	1.0899197	1.53992E-02	1.10531894	1.01097E-08	1.10531947	2.4996E-09	1.1053189529
0.5	1.1550367	2.19382E-02	1.17697495	1.26597E-08	1.176975821	4.0408E-09	1.1769749725
0.6	1.24443902	3.02400E-02	1.27467897	1.56044E-08	1.27468026	6.0642E-09	1.2746789919
0.7	1.36313979	4.08485E-02	1.40398829	1.95705E-08	1.403990155	8.7467E-09	1.403988318
0.8	1.51730030	5.44875E-02	1.57178774	2.57387E-08	1.57179035	1.2330E-08	1.5717877696

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0.9	1.71454198	7.21239E-0	1.78666581	3.62246E-08	1.78666945	1.7147E-08	1.78666585361
1.0	1.9643507	9.50567E-02	2.05940735	5.46971E-08	2.05941237	2.3655E-08	2.059407405

Table 3.Maximum Absolute error , exact $y(x$ and approximation solution for Example 1 with a uniform mesh size $h = 0.0$.

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	1.003973214	1.37331E-03	1.00534652	1.62280E-10	1.005346	5.3759E-12	1.005346521
0.2	1.019825416	3.06405E-03	1.02288946	3.21760E-10	1.0228894	3.1348E-11	1.0228894624
0.3	1.050026859	5.16510E-03	1.05519196	4.76790E-10	1.055191	6.4206E-11	1.0551919637
0.4	1.097520387	7.79857E-03	1.10531895	6.28600E-10	1.105318	1.0671E-10	1.1053189529
0.5	1.165849756	1.11252E-02	1.17697497	7.84350E-10	1.17697	1.6260E-10	1.1769749725
0.6	1.259321437	1.53576E-02	1.2746789	9.62520E-10	1.274679	2.3692E-10	1.2746789919
0.7	1.38321068	2.07776E-02	1.40398831	1.20166E-09	1.403988	3.3667E-10	1.403988318
0.8	1.544026207	2.77616E-02	1.57178776	1.57534E-09	1.571787	4.7152E-10	1.57178776967
0.9	1.749852424	3.68134E-02	1.78666585	2.21636E-09	1.78666	6.549E-10	1.78666585361
1.0	2.010795138	4.86123E-02	2.05940740	3.35651E-09	2.059407	9.0603 E-10	2.0594074053

Table 4.Maximum Absolute error , exact $y(x$ and approximation solution for Example 1 with a uniform mesh size h $= 0.01$.

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	1.00465666	6.89852E-04	1.005346521	1.01399E-11	1.00534652	4.405E-13	1.00534652181
0.2	1.0213494287	1.54003E-03	1.0228894624	2.00999E-11	1.0228894	1.2928E-12	1.02288946247
0.3	1.05259433197	2.59763E-03	1.0551919637	2.97600E-11	1.0551919	2.381E-12	1.05519196376
0.4	1.10139435479	3.92460E-03	1.105318952	3.91900E-11	1.10531895	3.8003E-12	1.10531895297
0.5	1.17137235329	5.60262E-03	1.176974972	4.88001E-11	1.17697497	5.6781E-12	1.1769749725
0.6	1.26693920489	7.73979E-03	1.2746789919	5.97400E-11	1.2746789	8.1932E-12	1.2746789919
0.7	1.39350856107	1.04798E-02	1.40398831	7.44000E-11	1.4039883	1.1588E-11	1.4039883184
0.8	1.557773406275	1.40144E-02	1.57178776957	9.73599E-11	1.5717877	1.6204E-11	1.57178776967
0.9	1.768064785719	1.86011E-02	1.7866658534	1.3693E-10	1.62071429	1.7601E-11	1.78666585361
1.00	2.0348201841	2.45872E-02	2.05940740513	2.0767E-10	2.05940743	3.12139E-11	2.0594074053

Table 5.Maximum Absolute error , exact $y(x$ and approximation solution for Example 2 with a uniform mesh size h= 0.

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	0.900000	1.35091E-02	0.91350893	1.95878E-07	0.913509	2.5187E-8	0.9135091278
0.2	0.82800000	2.12185E-02	0.84921817	3.47642E-07	0.84921810	2.7178E-07	0.8492185187
0.3	0.7760016	2.58218E-02	0.80182294	4.53096E-07	0.8018339	2.064E-07	0.8018233979
0.4	0.7390637996	2.87198E-02	0.76778306	5.24033E-07	0. 7677835	2.05454E-07	0.7677835861
0.5	0.7140048216	3.06849E-02	0.74468912	5.72232E-07	0.744689	2.91172E-07	0.7446897004
0.6	0.6987247742	3.21636E-02	0.73088779	6.06628E-07	0.7308884	3.06628E-07	0.7308884027
0.7	0.6918266296	3.34247E-02	0.72525066	6.33400E-07	0.725251	4.6013E-07	0.7252512992
0.8	0.6923920851	3.46350E-02	0.72702642	6.56635E-07	0.727027	6.30172E-07	0.7270270862
0.9	0.699842772	3.59008E-02	0.73574290	6.78965E-07	0.735743	6.44811E-07	0.7357435885
1.0	0.7138506309	3.72897E-02	0.75113964	7.02025E-07	0. 7511403	6.64912E-07	0.75114035195

Table 6.Maximum Absolute error , exact $y(x$ and approximation solution for Example 2 with a uniform mesh size h= 0..

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		
	y_n	e_r	y_n	e_n	y_n	y_r	$y(x_n)$
0.1	0.9072500	6.25913E-03	0.913509121	6.58113E-09	0.913530924	1.31842E-10	0.913509127
0.2	0.839260927	9.95759E-03	0.84921850	1.38536E-08	0.849218527	9.00325E-10	0.849218518
0.3	0.789588457	1.22349E-02	0.801823378	1.97359E-08	0.801823532	2.65418E-09	0.8018233979
0.4	0.754074326	1.37093E-02	0.767783562	2.41269E-08	767783586	5.61720E-09	0.7677835861
0.5	0.729957075	1.47326E-02	0.744689673	2.73825E-08	0.744686970	9.99691E-09	0.7446897004
0.6	0.715374409	1.55140E-02	0.730888372	2.98840E-08	0.730888401	1.60421E-08	0.7308884027
0.7	0.709069503	1.61818E-02	0.725251267	3.19353E-08	0.7252517	2.40821E-08	0.725251299
0.8	0.710209822	1.68173E-02	0.72702705	3.37550E-08	0.7270270	3.45600E-08	0.727027086
0.9	0.71827088	1.74727E-02	0.735743553	3.54931E-08	0.7357435	3.50716E-08	0.73574358
1.0	0.732958735	1.81816E-02	0.751140314	3.72487E-08	0.751140351	3.54106E-08	0.7511403519

Table 7.Maximum Absolute error , exact $y(x$ and approximation solution for Example 2 with a uniform mesh size h = 0.0.

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		
	y_n	e_r	y_n	e_r	y_n	e_r	$y(x_n)$
0.1	0.91048752	3.02160E-03	0.91350912	2.59029E-10	0.91350912	6.7543E-11	0.9135091278
0.2	0.84488477	4.83374E-03	0.84921851	6.51469E-10	0.84921851	4.6081E-11	0.8492185187
0.3	0.79585872	5.96467E-03	0.80182339	9.97784E-10	0.801823397	1.3588E-10	0.80182339795

0.4	0.761077717	6.70587E-03	0.76778358	1.26955E-09	0.767783589	2.8763E-10	0.76778358615
0.5	0.737463994	7.22571E-03	0.74468969	1.47866E-09	0.744689700	5.1191E-10	0.74468970047
0.6	0.723263149	7.62525E-03	0.73088840	1.64401E-09	0.730888402	8.2158E-10	0.7308884027
0.7	0.717284069	7.96723E-03	0.72525129	1.78223E-09	0.725251295	1.2333E-09	0.72525129927
0.8	0.718735476	8.29161E-03	0.72702708	1.90588E-09	0.7270285	1.7700E-09	0.72702708621
0.9	0.727119314	8.62427E-03	0.73574358	2.02387E-09	0.73574358	2.0062E-09	0.73574358854
1.0	0.742158513	8.98184E-03	0.75114034	2.14228E-09	7.511403	2.1190E-09	0.75114035195

Table 8.Maximum Absolute error , exact $y(x$ and approximation solution for Example 2 with a uniform mesh size h $= 0.01$.

x_n	Previous methods				Present Method		Exact solution
	Euler Method		Runge-Kutta method		Adams-Mutton methods		$y(x_n)$
	y_n	e_r	y_n	e_r	y_n	e_r	
0.1	0.9120236443	1.48548E-03	0.913509127	1.17910E-11	0.91350912	3.41816E-12	0.9135091278
0.2	0.846835976	2.38254E-03	0.849218518	3.44851E-11	0.84921851	2.3322E-12	0.8492185187
0.3	0.798877481	2.94592E-03	0.801823397	5.54771E-11	0.80182339	5.08776E-11	0.8018233979
0.4	0.7644662941	3.31729E-03	0.767783586	7.23600E-11	0.76778358	6.45588E-11	0.7677835861
0.5	0.74111067	3.57903E-03	0.744689700	8.55770E-11	7.44689700	7.59133E-11	0.7446897004
0.6	0.727107563	3.78084E-03	0.730888402	9.61640E-11	0.7308884	9.41586E-11	0.7308884027
0.7	0.7212975927	3.95371E-03	0.725251299	1.05089E-10	0.7252512	1.03289E-10	0.7252512992
0.8	0.7229096345	4.11745E-03	0.727027086	1.13103E-10	0.72702708	1.09958E-10	0.7270270862
0.9	0.7314586485	4.28494E-03	0.73574358	1.20751E-10	0.73574358	1.14154E-10	0.7357435885
1.0	0.7466759122	4.46444E-03	0.751140351	1.28405E-10	0.75114035	1.19346E-10	0.7511403519

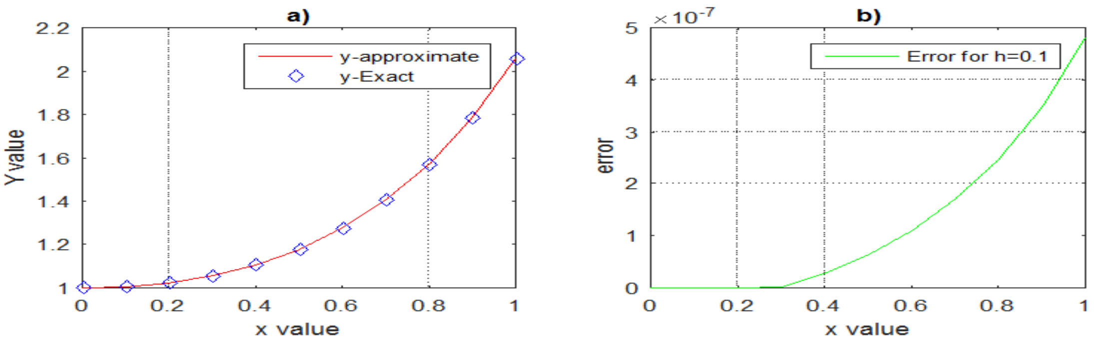

Figure 1: Geometric representation of solution and maximum error for example 1 with space size h=0.1

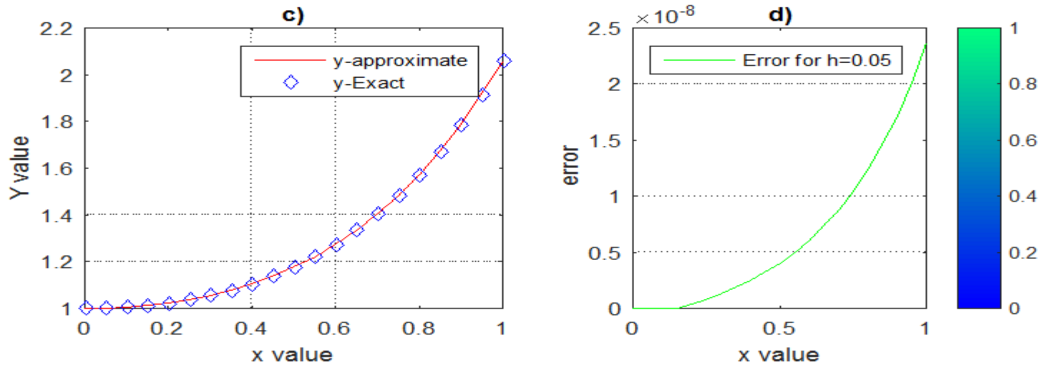

Figure 2: Geometric representation of solution and maximum error for example 1 with space size h=0.05

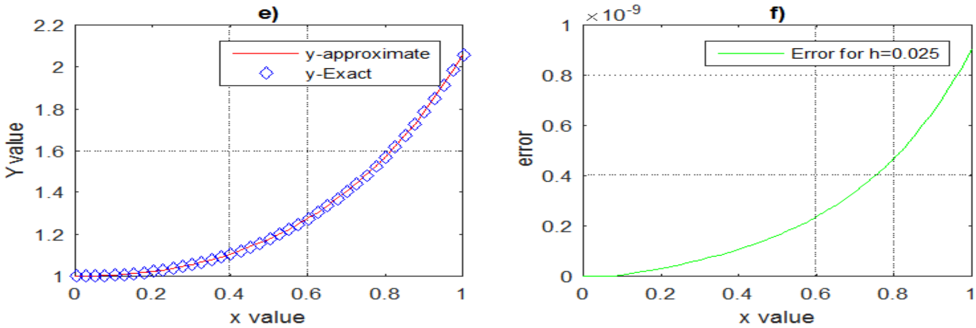

Figure 3: Geometric representation of solution and maximum error for example 1 with space size h=0.025

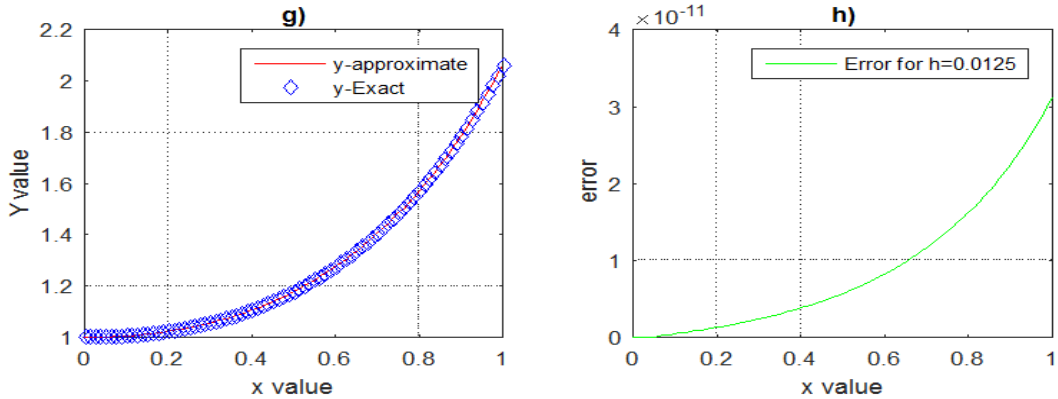

Figure 4: Geometric representation of solution and maximum error for example 1 with space size h=0.0125

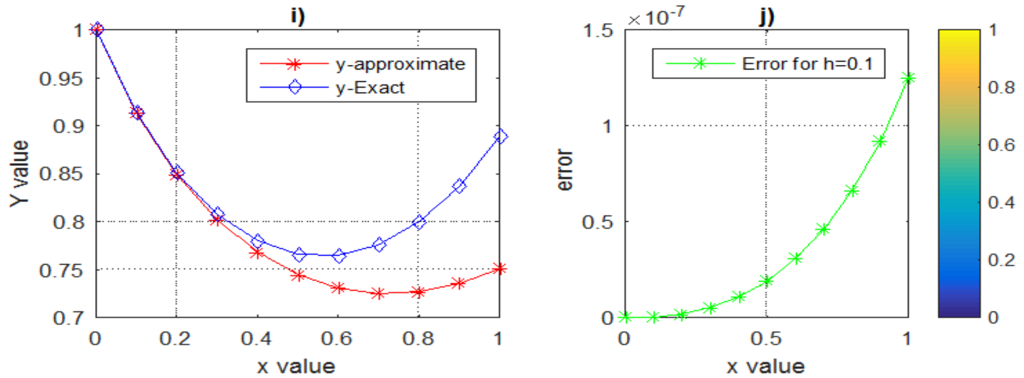

Figure 5: Geometric representation of solution and maximum error for example 2 with space size h=0.1

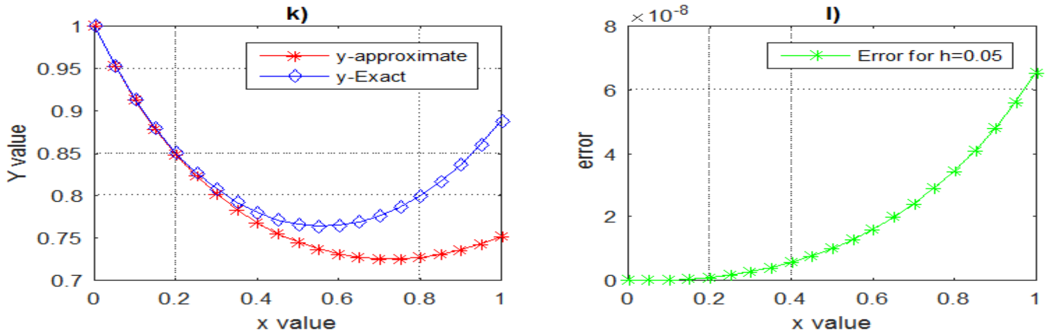

Figure 6: Geometric representation of solution and maximum error for example 2 with space size h=0.05

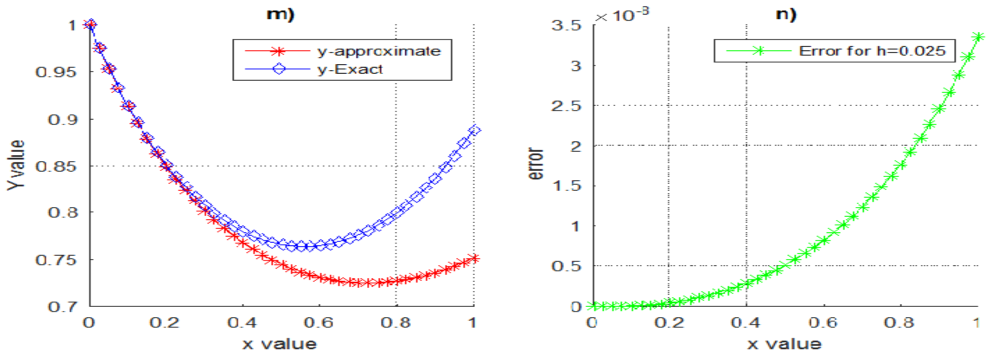

Figure 7: Geometric representation of solution and maximum error for example 2 with space size h=0.025

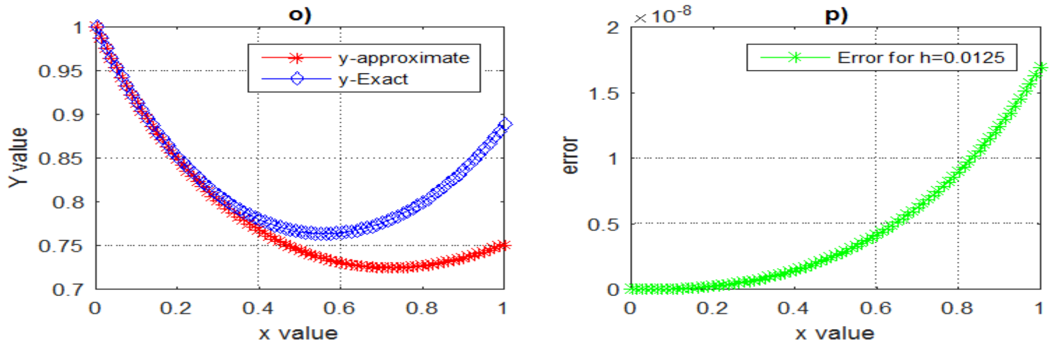

Figure 8: Geometric representation of solution and maximum error for example 2 with different space size

VI. DISCUSSION AND CONCLUSION

In this paper, we described the fourth-order Adams predictor corrector method for solving initial value problems. To demonstrate the competence of the method, we applied it on two model examples by taking different values for mesh size, h. Numerical results obtained by the present method have been associated with numerical results obtained by the methods in [1] and the results are summarized in Tables 1-8 and graph. As can be seen from the numerical results and predicted in tables 1-8 and graphs 1-8 above, the present method is superior to the method developed in [1] and approximate the exact solution very well. Moreover, the maximum absolute errors decrease rapidly as the number of mesh point's N increases.

Further, as shown in Figs. 1-8, the proposed method approximates the exact solution very well for which most of the current methods fail to give good results.

To further verify the applicability of the planned method, graphs were plotted aimed at Examples 1 and 2 for exact solutions versus the numerical solutions obtained with their errors. As Figs. 1-4 indicate good agreement of the results, presenting exact as well as numerical solutions, which proves the reliability of the method for example 1. Also, Figs. 5 - 8 indicate good agreement of the results for example 2 with different mesh sizes of the solution domain. Further, the numerical results presented in this paper validate the improvement of the proposed method over some of the existing methods described in the literature. Both the theoretical and numerical error bounds have been established for the fourth order Adams predictor corrector methods.

Abbreviations

IVP initial value problem

e_r Maximum absolute error

Eq. Equation.

1. Declaration

Does not applicable

Availability of Data and Materials

All data generated or analyzed during this study are included.

Competing interests

The authors declare that they have no competing interests.

Funding

Not applicable.

Authors' Contributions

'KA' proposed the main idea of this paper. 'KA' and 'GK' prepared the manuscript and performed all the steps of the proofs in this research. Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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