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Special Diophantine Triples Involving Square Pyramidal Numbers

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***Abstract:** In this communication, we accomplish special Diophantine triples comprising of square pyramidal numbers such that the product of any two members of the set added by their sum and increased by a polynomial with integer coefficient is a perfect square*

***Keywords:** Special Diophantine Triples, Square Pyramidal Number, Perfect Square.*

I. INTRODUCTION

Number theory is fascinating on the grounds that it has such a large number of open problems that seem accessible from the outside. Of course, open problems in number theory are open for a reason. Numbers, despite being simple, have an incredibly rich structure which we only understand to a limited degree. In the mid twentieth century, Thue made an important breakthrough in the study of Diophantine equations. His proof is one of the polynomial methods His proof impacted a great deal of later work in number theory, including Diophantine equations. Various mathematicians considered the problem of the existence of Diophantine triples with the property D(n) for any integer n and besides for any linear polynomial in n [1-5]. Right now, one may suggest for an extensive survey of different issues on Diophantine triples[6-7]. In [8-9], square pyramidal numbers were evaluated using Z-transform and division algorithm. In [10-12], Diophantine triples were discussed. In this paper, we exhibit special Diophantine triples (a, b, c) involving square pyramidal number such that the product of any two elements of the set added by their sum and increased by a polynomial with integerco-efficient is a perfect square.

II. NOTATION

p_n^4 : square pyramidal number of rank n.

III. BASIC DEFINITION

A set of three different polynomials with integer coefficients (a_1, a_2, a_3) is said to be a special Diophantine triple with property.

D(n) if $a_i * a_j + (a_i + a_j) + n$ is a perfect square for all $1 \le i < j \le 3$,where n may be non-zero integer or polynomial with integer coefficients.

IV. METHOD OF ANALYSIS

A. *Construction of the special dio-3 triples involving square pyramidal number of rankn and n − 1*

Let $a = 6p_n^4$ and $b = 6p_{n-1}^4$ be square pyramidal numbers of rank n and $n-1$ respectively.

Now,$a = 6p_n^4$ and $b = 6p_{n-1}^4$

$$ab + (a+b) + n^4 - 4n + 1$$
$$= 4n^6 - 4n^4 + 4n^3 + n^2 - 2n + 1$$
$$= (2n^3 - n+1)^2 = \alpha^2 \quad (1)$$

Equation (1) is a perfect square.

$ab + (a+b) + n^4 - 4n + 1 = \alpha^2$ where $\alpha = 2n^3 - n + 1$

Let c be non zero-integer such that,

$$ac + (a+c) + n^4 - 4n + 1 = \beta^2 \quad (2)$$

$$bc + (b+c) + n^4 - 4n + 1 = \gamma^2 \quad (3)$$

Solving (2) & (3)⇒ $c(b-a) + (b-a) = b\beta^2 - a\gamma^2$ (4)

(3) -(2) ⇒ $\gamma^2 - \beta^2 = c(b-a) + b - a$

Therefore (4) becomes,

$$\gamma^2 - \beta^2 = b\beta^2 - a\gamma^2$$
$$(b+1)\beta^2 - (a+1)\gamma^2$$

Setting $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y$,

$$\Rightarrow (b+1)(x + (a+1)y)^2 - (a+1)(x + (b+1)y)^2 \quad (5)$$

Now put $y = 1$,

$$x^2 = (2n^3 - n + 1)^2$$
$$\Rightarrow x = (2n^3 - n + 1)$$

The initial solution of (5) is given by,

$x_0 = (2n^3 - n + 1), y_0 = 1$

Since, $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y$, we obtain that,

$\beta = 4n^3 + 3n^2 + 2$

Therefore, the equation (2) becomes,

$$(2) \Rightarrow ac + (a+c) + n^4 - 4n + 1 = \beta^2$$
$$\Rightarrow c(a+1) + a + n^4 - 4n + 1 = \beta^2$$
$$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) + n^4 - 4n + 1 = \beta^2$$
$$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) + n^4 - 4n + 1 = (4n^3 + 3n^2 + 2)^2$$
$$\Rightarrow c = 8n^3 + 3$$
$$\Rightarrow c = (2(a + b - 4n + 3))$$

Therefore, the triples

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$\{a, b, (2(a+b-4n+3))\}$
$= \{6p_n^4\ ,\ 6p_{n-1}^4, (2(6p_n^4 + 6p_{n-1}^4 - 4n + 3))\}$ is a Diophantine triples with the property $D(n^4-4n+1)$.
Some numerical examples are given below in the following table.

Table 1

n	Diophantine Triples	$D(n^4-4n+1)$
1	(6,0, 11)	-2
2	(30,6,67)	9
3	(84,30,219)	70

B. Construction of the special dio-3 triples involving square pyramidal number of rankn and $n-2$

Let $a = 6p_n^4$ and $b = 6p_{n-2}^4$ be square pyramidal numbers of rank n and $n-2$respectively.
Now, $a = 6p_n^4$ and $b = 6p_{n-2}^4$
$ab + (a+b) - 2n^3 + 3n^2 - 16n + 10$
$= 4n^6 - 12n^5 + n^4 + 20n^3 - 8n^2 - 8n + 4$
$= (2n^3 - 3n^2 - 2n{+}2)^2 = \alpha^2(6)$
Equation (6) is a perfect square.
$ab + (a+b) - 2n^3 + 3n^2 - 16n + 10 = \alpha^2,$
where $\alpha = 2n^3 - 3n^2 - 2n + 2$
Let c be non zero-integer such that,
$ac + (a+c) - 2n^3 + 3n^2 - 16n + 10 = \beta^2$ (7)
$bc + (b+c) - 2n^3 + 3n^2 - 16n + 10 = \gamma^2$ (8)
Solving (7) & (8)$\Rightarrow c(b-a) + (b-a) = b\beta^2 - a\gamma^2(9)$
(8) -(7)$\Rightarrow \gamma^2 - \beta^2 = c(b-a) + b - a$
Therefore (9) becomes,

$$\gamma^2 - \beta^2 = b\beta^2 - a\gamma^2$$
$$(b+1)\beta^2 - (a+1)\gamma^2$$

Setting $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y$
$\Rightarrow (b+1)(x+(a+1)y)^2 - (a+1)(x + (b + 1y)2(10)$
Now put $y = 1,$

$$x^2 = (2n^3 - 3n^2 - 2n{+}2)^2$$
$$\Rightarrow x = (2n^3 - 3n^2 - 2n + 2)$$

The initial solution of (10) is given by,
$x_0 = (2n^3 - 3n^2 - 2n + 2)\ , y_0 = 1$
Since, $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y,$ we obtain that,

$$\beta = 4n^3 - n + 3$$

Therefore, the equation (7) becomes,
(7)$\Rightarrow ac + (a+c) - 2n^3 + 3n^2 - 16n + 10 = \beta^2$
$\Rightarrow c(a+1) + a - 2n^3 + 3n^2 - 16n + 10 = \beta^2$
$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) - 2n^3 + 3n^2 - 16n + 10 = \beta^2$
$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) - 2n^3 + 3n^2 - 16n + 10 = (4n^3 - n + 3)^2$
$\Rightarrow c = 8n^3 - 12n^2 + 10n - 1$
$\Rightarrow c = (2(a+b) - 18n + 11)$
Therefore, the triples
$\{a, b, (2(a+b)) - 18n + 11))\}$
$= \{6p_n^4\ , 6p_{n-2}^4, (2(6p_n^4 + 6p_{n-2}^4) - 18n + 11)\}$ is a Diophantine triples with the $D(-2n^3 + 3n^2 - 16n + 10)$.
Some numerical examples are given below in the following table.

Table 2

n	Diophantine Triples	$D(-2n^3 + 3n^2 - 16n + 10)$
1	(6,0, 5)	-5
2	(30,0,35)	-26
3	(84,6,137)	-65

C. Construction of the special dio-3 triples involving square pyramidal number of rankn and $n-3$

Let $a = 6p_n^4$ and $b = 6p_{n-3}^4$ be square pyramidal numbers of rank n and $n-3$ respectively.
Now,$a = 6p_n^4$ and $b = 6p_{n-3}^4$
$ab + (a+b) + n^4 - 18n^2 - 22n + 7$
$= 4n^6 - 24n^5 + 32n^4 + 40n^3 - 83n^2 - 14 + 49$
$= (2n^3 - 6n^2 - n{+}7)^2 = \alpha^2(11)$
Equation (11) is a perfect square.
$ab + (a+b) + n^4 - 18n^2 - 22n + 79 = \alpha^2,$
where $\alpha = 2n^3 - 6n^3 - n + 7$
Let c be non zero-integer such that,
$ac + (a+c) + n^4 - 18n^2 - 22n + 79 = \beta^2$ (12)
$bc + (b+c) + n^4 - 18n^2 - 22n + 79 = \gamma^2$ (13)
Solving (12) & (13)$\Rightarrow c(b-a) + (b-a) = b\beta^2 - a\gamma^2(14)$
(13) -(12)$\Rightarrow \gamma^2 - \beta^2 = c(b-a) + b - a$
Therefore (14) becomes,

$$\gamma^2 - \beta^2 = b\beta^2 - a\gamma^2$$
$$(b+1)\beta^2 - (a+1)\gamma^2$$

Setting $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y,$
$\Rightarrow (b+1)(x + (a+1)y)^2 - (a+1)(x + (b+1)y)^2(15)$
Now put $y = 1,$

$$x^2 = (2n^3 - 6n^2 - n{+}7)^2$$
$$\Rightarrow x = (2n^3 - 6n^2 - n + 7)$$

The initial solution of (15) is given by,
$x_0 = (2n^3 - 6n^2 - n + 7)\ , y_0 = 1$
Since, $\beta = x + (a+1)y$ and $\gamma = x + (b+1)y,$ we obtain that,

$$\beta = 4n^3 - 3n^2 + 8$$

Therefore, the equation (12) becomes,
(12)$\Rightarrow ac + (a+c) + n^4 - 18n^2 - 22n + 79 = \beta^2$
$\Rightarrow c(a+1) + a + n^4 - 18n^2 - 22n + 79 = \beta^2$
$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) + n^4 - 18n^2 - 22n + 79 = \beta^2$
$\Rightarrow c(2n^3 + 3n^2 + n + 1) + (2n^3 + 3n^2 + n) + n^4 - 18n^2 - 22n + 7 = (4n^3 - 3n^2 + 8)^2$
$\Rightarrow c = 8n^3 - 24n^2 + 36n - 15$
$\Rightarrow c = (2(a+b) - 40n + 45)$
Therefore, the triples
$\{6p_n^4\ ,\ 6p_{n-3}^4, (2(6p_n^4 + 6p_{n-3}^4) - 40n + 45)\}$ is a Diophantine triples with the property $D(n^4 - 18n^2 - 22n + 79)$.
Some numerical examples are given below in the following table.

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Table 3

n	Diophantine Triples	$D(n^4 - 18n^2 - 22n + 79)$
1	(6, -6, 5)	40
2	(30,0,25)	-21
3	(83,0,93)	-68

V. CONCLUSION

We have presented the special Diophantine triples involving square pyramidal numbers. To conclude one may look for triples or quadruples for different numbers with their relating properties.

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