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Hardy-Littlewood-Type Theorem for Mixed Fractional Integrals in Hölder Spaces

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Abstract*:** ***We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained are results generalized to the case of Hölder spaces with power weight

Keywords*:** ***functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, weight, mixed fractional integral, Hölder space.

I. INTRODUCTION

In1928 H.G.Hardy and J.E.Littlewood [1] (see [13], Theorem 3.1 and 3.2) showed that the fractional integral

$$\left(I^{\alpha}_{0+}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)dt}{(x-t)^{1-\alpha}},\quad 0<x<1$$

of order $\alpha\in(0,1)$ improves the Hölder behavior of its density exactly by the order α. More exactly, these operators establish an isomorphism between the spaces $H^{\lambda}_0([0,1])$ and $H^{\lambda+\alpha}_0([0,1])$ under the condition $\alpha+\lambda$. This result was extended in many directions: to the case of Hölder spaces with power weight [15] to the case of generalized Hölder spaces with characteristics from the Bari-Stechkin class [13], [14]; to the case of more general weights [16], [17], etc. Different proofs were suggested in [2], [3], where the case of complex fractional orders was also considered the shortest proof being given in [2].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann - Liouville integral was studied in [4]-[12].

As is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between weighted Hölder spaces for functions one variable. But for functions two variable were not studied. This paper is aimed to fill in this gap. We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained results generalized to the case of Hölder spaces with power weight of functions of two variables.

Mixed Riemann-Liouville fractional integrals of order (α,β)

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}},\qquad(1)$$

$x>0,y>0$.

Mixed fractional derivatives form Marchaud of order (α,β)

$$\left(\mathrm{D}^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)=\frac{\varphi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}x^{-\alpha}y^{-\beta}+$$

$$+\frac{\alpha\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(x,y)-\varphi(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds,\ x>0,y>0$$

(2)

Consider the operators in a rectangle $Q=\{(x,y):0<x<b,0<y<d\}$.

II. PRELIMINARY

For a continuous function $\varphi(x,y)$ on R^2 we introduce the notation

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),$$

$$\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y).$$

So that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)+\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+$$

$$+\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)+\varphi(x,y).\qquad(3)$$

Everywhere in the sequel by C,C_1,C_2 etc we denote positive constants which may have different values in different occurrences and even in the same line.

Definition 1. Let $\lambda,\gamma\in(0,1]$. We say that $\varphi(x,y)\in H^{\lambda,\gamma}(Q)$, if

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)|\le C_1|x_1-x_2|^{\lambda}+C_2|y_1-y_2|^{\gamma}\qquad(4)$$

for all $(x_i,y_i)\in Q, i=1,2$.

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Condition (4) is equivalent to the couple of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right| \le C_1 |h|^{\lambda}, \quad \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right| \le C_2 |\eta|^{\gamma}$$

uniform with respect to another variable.

By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $\varphi \in H^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

Let $\lambda = 0$ and\or $\gamma = 0$. We put $H^{0,0}(Q) = L^\infty(Q)$ and

$$H^{\lambda,0}(Q) = \left\{\varphi \in L^\infty(Q) : \left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right| \le C_1 |h|^{\lambda}\right\}, \ \lambda \in (0,1]$$

$$H^{0,\gamma}(Q) = \left\{\varphi \in L^\infty(Q) : \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right| \le C_2 |\eta|^{\gamma}\right\}, \ \gamma \in (0,1].$$

Definition 2. We say that $\varphi(x,y) \in \tilde{H}^{\lambda,\gamma}(Q)$, where $\lambda, \gamma \in (0,1]$, if

$$\varphi \in H^{\lambda,\gamma}(Q) \text{ and } \left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| \le C_3 |h|^{\lambda} |\eta|^{\gamma}.$$

We say that $\varphi \in \tilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi \in H^{\lambda,\gamma}(Q)$ and $\varphi(0,y) \equiv \varphi(x,0) \equiv 0$.

These spaces become Banach spaces under the standard definition of thenorms:

$$\|\varphi\|_{H^{\lambda,\gamma}} := \|\varphi\|_{C(Q)} + \sup_{\substack{x,x+h\in[0,b]\\ y\in[0,d]}} \frac{\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right|}{|h|^{\lambda}} + \sup_{\substack{y,y+\eta\in[0,b]\\ x\in[0,b]}} \frac{\left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right|}{|\eta|^{\gamma}},$$

$$\|\varphi\|_{\tilde{H}^{\lambda,\gamma}} := \|\varphi\|_{H^{\lambda,\gamma}} + \sup_{\substack{x,x+h\in[0,b]\\ y,\eta+y\in[0,d]}} \frac{\left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right|}{|h|^{\lambda} |\eta|^{\gamma}}.$$

Note that

$$\varphi(x,y) \in H^{\lambda,\gamma}(Q) \Rightarrow \left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| \le C_\theta |h|^{\theta\lambda} |\eta|^{(1-\theta)\gamma} \tag{5}$$

for any $\theta \in [0,1]$, where $C_\theta 2 = C_1^{\theta} C_2^{1-\theta}$.

Proof. Let $|h|^{\lambda} \ge |\eta|^{\gamma}$, then

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| = \left| \varphi(x+h, y+\eta) - \varphi(x, y+\eta) - \right.$$
$$\left. - \varphi(x+h, y) + \varphi(x,y) \right| \le$$
$$\le \left| \varphi(x+h, y+\eta) - \varphi(x+h, y) \right| +$$
$$+ \left| \varphi(x, y+\eta) - \varphi(x,y) \right| \le C |\eta|^{\gamma} =$$
$$= C |\eta|^{\gamma[\theta + (1-\theta)]} \le C_\theta |h|^{\theta\lambda} |\eta|^{(1-\theta)\gamma}.$$

Case $|h|^{\lambda} \le |\eta|^{\gamma}$ is considered similar.

So that

$$\bigcap_{0\le\theta\le1} \tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q) \hookrightarrow H^{\lambda,\gamma}(Q) \hookrightarrow \tilde{H}^{\lambda,\gamma}(Q) \tag{6}$$

where $\hookrightarrow$ stands for the continuous embedding.

The norm for $\bigcap_{0\le\theta\le1} \tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q)$. Since $\theta \in [0,1]$ is arbitrary, it is not hard to see that the inequality in (5) is equivalent (up to the constant factor *C*) to

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| \le C \min\left\{|h|^{\lambda}, |\eta|^{\gamma}\right\}. \tag{7}$$

We will also make use of the following weighted spaces. Let $\rho(x,y)$ be a non-negative function on Q (we will only deal with degenerate weights $\rho(x,y) = \rho_1(x)\rho_2(y)$).

Definition 3. By $H^{\lambda,\gamma}(Q,\rho)$ and $\tilde{H}^{\lambda,\gamma}(Q,\rho)$ we denote the spaces of functions $\varphi(x,y)$ such that $\rho\varphi \in H^{\lambda,\gamma}(Q)$ and $\rho\varphi \in \tilde{H}^{\lambda,\gamma}(Q)$ respectively, equipped with the norms

$$\|\varphi\|_{H^{\lambda,\gamma}(Q,\rho)} = \|\rho\varphi\|_{H^{\lambda,\gamma}(Q)} \text{ and } \|\varphi\|_{\tilde{H}^{\lambda,\gamma}(Q,\rho)} = \|\rho\varphi\|_{\tilde{H}^{\lambda,\gamma}(Q)}.$$

By $H_0^{\lambda,\gamma}(Q,\rho)$ and $\tilde{H}_0^{\lambda,\gamma}(Q,\rho)$ we denote the corresponding subspaces of functions φ such that $\rho\varphi|_{x=0} = \rho\varphi|_{y=0} \equiv 0$.

Below we follow some technical estimations suggested in [2] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x,y;t,s) = \frac{\rho(x,y) - \rho(t,s)}{\rho(t,s)(x-t)^{\delta}(y-s)^{\sigma}}, \tag{8}$$

where $0 < \delta, \sigma < 2$, $a < t < x < b$, $0 < s < y < d$ and

$$B_1(x,t) = \frac{\rho_1(x) - \rho_1(t)}{\rho_1(t)(x-t)^{\delta}}, \quad B_2(y,s) = \frac{\rho_2(y) - \rho_2(s)}{\rho_2(s)(y-s)^{\sigma}}. \tag{9}$$

In the case $\rho(x,y) = \rho_1(x)\rho_2(y)$ we have

$$B(x,y;t,s) = B_1(x,t)B_2(y,s) + \frac{B_1(x,t)}{(y-s)^{\sigma}} + \frac{B_2(y,s)}{(x-t)^{\delta}}.$$

Let also

$$D_1(x,h,t) = B_1(x+h,t) - B_1(x,t), \ t, x, x+h \in [0,b], \ h > 0,$$
$$D_2(y,\eta,s) = B_2(y+\eta,s) - B_2(y,s), \ s, y, y+\eta \in [0,d], \ \eta > 0.$$

Lemma 1 ([2]). Let $\rho_1(x) = x^{\mu}$, $\mu \in R^1$ and $\delta = 1-\alpha, \alpha \in (0,1)$. Then

$$|B_1(x,t)| \le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)} \frac{(x-t)^{\alpha}}{t}, \tag{10}$$

$$|D_1(x,t,h)| \le C\left(\frac{x+h}{t}\right)^{\max(\mu-1,0)} \frac{h}{t(x+h-t)^{1-\alpha}}. \tag{11}$$

Lemma 2 ([2]). Let $\rho_1(x) = x^{\mu}$, $\mu \in R^1$ and $\delta = 1+\alpha, \alpha \in (0,1)$. Then

$$|B_1(x,t)| \le C\left(\frac{x}{t}\right)^{\max(\mu-1,0)} \frac{1}{t(x-t)^{\alpha}}, \tag{12}$$

$$|D_1(x,t,h)| \le C\left(\frac{x+h}{t}\right)^{\max(\mu-1,0)} \frac{h}{t(x+h-t)(x-t)^{\alpha}}. \tag{13}$$

Similar estimates hold for $B_2(y,s)$ and $D_2(y,\eta,s)$ with $\rho(y) = y^{\mu}$.

Remark. All the weighted estimations of fractional integrals in the sequel are based on inequalities (10)-(13). Note that the right-hand sides of these inequalities have the exponent $\max(\mu-1,0)$, which means that in the proof it suffices to consider only the case $\mu \ge 1$, evaluations for $\mu < 1$ being the same as for $\mu = 1$.

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The following statement is known (see the presentation of this proof in [16], p. 190); a shorter proof was given in [2]. Nevertheless we recall the scheme of the proof from [2] to make the presentation easier for the two-dimensional case.

Let $\rho(x)$ the weight function and put $\psi(x)=\rho(x)\varphi(x)$, $\varphi\in H_0^\lambda([0,b];\rho)$. Evidently $\psi(x)\in H_0^\lambda([0,b])$ and $\psi(0)=0$. It is easy to see that

$$\left(\rho I_{0+}^\alpha\varphi\right)(x)=\left(I_{0+}^\alpha\psi\right)(x)+\left(M_{0+}^\alpha\psi\right)(x), \qquad (14)$$

where

$$\left(M_{0+}^\alpha\psi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x B_1(x,t)\psi(t)dt,$$

so that in (14) we have the fractional integral if $0<\alpha<1$ and the fractional derivative if $-1<\alpha<0$.

The representation (14) for the fractional (integral) derivative shows that the estimate for the continuity modulus in the weighted case reduces to two simpler estimates:

1) the known non-weighted estimate of Hardy-Littlewood for fractional integral and fractional derivative;

2) the estimate of the second term in (14), which is the main part of the job.

Theorem 1. Let $0<\alpha<1$, $\rho(x)=x^\mu$ and $|\psi(x)|\le Cx^{\alpha+\lambda}$ with $\mu<1+\lambda$. Then operator $\left(M_{0+}^{-\alpha}\psi\right)(x)\in H^\lambda([0,b])$, $\lambda+\alpha<1$ and

$$\left\|\left(M_{0+}^{-\alpha}\psi\right)(x)\right\|_{H^\lambda}\le C\sup_{x\in[0,b]}\left|\frac{\psi(x)}{x^{\alpha+\lambda}}\right|.$$

Proof. In the proof, we use the following notation

$$\left(M_{0+}^{-\alpha}\psi\right)(x+h)-\left(M_{0+}^{-\alpha}\psi\right)(x)=F_1(x,h)+F_2(x,h),$$

where

$$F_1(x,h)=\int_x^{x+h}B_1(x+h,t)\psi(t)dt,$$

$$F_2(x,h)=\int_0^x D_1(x,h,t)\psi(t)dt.$$

The estimate of F_1 in the case $1\le\mu<1+\lambda$. The estimate (12) for $B_1(x,t)$ implies

$$|F_1|\le C(x+h)^{\mu-1}\int_x^{x+h}\frac{t^{\lambda+\alpha-\mu}dt}{(x+h-t)^\alpha}=$$
$$=C(x+h)^\lambda\int_0^{\frac{h}{x+h}}\xi^{-\alpha}(1-\xi)^{\lambda+\alpha-\mu}d\xi\le C_1h^\lambda,$$

where

$$C_1=C\sup_{0<\varepsilon<1}\frac{1}{\varepsilon^\lambda}\int_0^\varepsilon\xi^{-\alpha}(1-\xi)^{\lambda+\alpha-\mu}d\xi<\infty.$$

The estimate of F_2 in the case $1\le\mu<1+\lambda$. Applying the estimate (13) for $D_1(x,h,t)$, we obtain

$$|F_2|\le C(x+h)^{\mu-1}\int_x^{x+h}\frac{t^{\lambda+\alpha-\mu}dt}{(x+h-t)(x-t)^\alpha}=$$
$$=Ch(x+h)^{\lambda-1}\int_0^{\frac{x+h}{h}}\frac{\xi^{\lambda+\alpha-\mu}d\xi}{(1-\xi)\left(\frac{x}{x+h}-\xi\right)^\alpha}\le C_2h^\lambda,$$

where

$$C_2=C\sup_{0<\varepsilon<1}\varepsilon^{1-\lambda}\int_0^{1-\varepsilon}\frac{\xi^{\lambda+\alpha-\mu}d\xi}{(1-\xi)(1-\varepsilon-\xi)^\alpha}<\infty.$$

III. MAIN RESULT

The definition in the Marchaud form may be used for all $-1<\alpha,\beta<1$: if $\alpha,\beta>0$ (2) gives the mixed fractional derivative, if $\alpha,\beta<0$, it is mixed fractional integral.

Let $\rho(x,y)=\rho(x)\rho(y)$ be the weight function and put $\psi(x,y)=\rho(x,y)\varphi(x,y)$ $\varphi\in\tilde{H}_0^{\lambda,\gamma}(Q;\rho)$. Evidently $\psi\in\tilde{H}^{\lambda,\gamma}(Q)$ and $\psi(x,y)\big|_{x=0,y=0}=0$. It is easy to see that

$$\left(\rho J_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\left(J_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)+C\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y), \qquad (15)$$

where $-1<\alpha,\beta<1$, C=const and

$$\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y)=\int_0^x\int_0^y B(x,y;t,s)\psi(t,s)dtds,$$

so that in (15) we have the mixed fractional integral if $0\le\alpha,\beta<1$ and the mixed fractional derivative if $-1<\alpha,\beta\le0$.

The representation (15) for the fractional (integral) derivative shows that the estimate for the continuity modulus in the weighted case reduces to two simpler estimates:

1) the known non-weighted estimate of Hardy-Littlewood for mixed fractional integral (see [4]) and mixed fractional derivative (see [7]); in the case weighted estimated of Hardy-Littlewood for mixed fractional integral (see [4]);

2) the estimate of the second term in (15), which is the mixed fractional derivative. It is main part of the job.

Theorem 2. Let $0<\alpha,\beta<1$, $0<\lambda,\gamma<1$, $\rho(x,y)=x^\mu y^\nu$ and let $\alpha+\lambda<1$, $\beta+\gamma<1$ and $|\psi(x,y)|\le Cx^{\lambda+\alpha}y^{\gamma+\beta}$ with $\mu<1+\lambda$, $\nu<1+\gamma$. Then the operator $K_{0+,0+}^{\alpha,\beta}$ is bounded from the space $\tilde{H}_0^{\lambda,\gamma}(\rho)$ into $\tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(\rho)$.

Proof. We don't proof this theorem. The proof of this theorem can be seen in [3].

Theorem 3. Let $0<\alpha,\beta<1$, $0<\lambda,\gamma<1$, $\rho(x,y)=x^\mu y^\nu$ and let $\alpha+\lambda<1$, $\beta+\gamma<1$ and $|\psi(x,y)|\le Cx^\lambda y^\gamma$ with $\mu<1+\lambda$, $\nu<1+\gamma$. Then the operator $K_{0+,0+}^{-\alpha,-\beta}$ is bounded from the space $\tilde{H}_0^{\lambda,\gamma}(\rho)$ into $\tilde{H}_0^{\lambda-\alpha,\gamma-\beta}(\rho)$.

Proof. To estimate the term $\left(K_{0+,0+}^{-\alpha,-\beta}\psi\right)(x,y)$, we note that the weight being degenerate, we have

$$\begin{aligned}\rho(x,y)-\rho(t,s)&=[\rho(x)-\rho(t)][\rho(y)-\rho(s)]+\\&+\rho(s)[\rho(x)-\rho(t)]+\rho(t)[\rho(y)-\rho(s)]\end{aligned}.$$

This leads to the following representation

$$\left(K_{0+,0+}^{-\alpha,-\beta}\psi\right)(x,y)=G_2(x,y)=\int_0^x\int_0^y B_1(x,t)B_2(y,s)\psi(t,s)dtds+$$
$$+\int_0^x\int_0^y B_1(x,t)\frac{\psi(t,s)}{(y-s)^{1+\beta}}dtds+\int_0^x\int_0^y B_2(y,s)\frac{\psi(t,s)}{(x-t)^{1+\alpha}}dtds,$$

here we used the notation (9).

For $h>0$ and $x,x+h\in(0,b)$, we consider the difference

$$G_2(x+h,y)-G_2(x,y)=\int_x^{x+h}\int_0^y B_1(x+h,t)B_2(y,s)\psi(t,s)dtds+$$

$$+\int_0^x\int_0^y D_1(x,h,t)B_2(y,s)\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\frac{\psi(t,s)}{(y-s)^{1+\beta}}dtds+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\frac{\psi(t,s)}{(y-s)^{1+\beta}}dtds+$$

$$+\int_x^{x+h}\int_0^y B_2(y,s)\frac{\psi(t,s)}{(x+h-t)^{1+\alpha}}dtds+$$

$$+\int_0^x\int_0^y B_2(y,s)\psi(t,s)\left[\,(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right]dtds.$$

Since $\psi\in\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}$, we have

$$|\psi(t,s)|\le Ct^{\lambda+\alpha}s^{\gamma+\beta},$$
$$|\psi(t,s)-\psi(x,0)|\le C(t-x)^{\lambda+\alpha}s^{\gamma+\beta}. \qquad (16)$$

Use (16) and (13) for variable *s*, we obtained

$$|G_2(x+h,y)-G_2(x,y)|\le C\Bigg\{\Bigg(\int_x^{x+h}|B_1(x+h,t)|t^{\lambda+\alpha}dt+$$

$$+\int_0^x|D_1(x,h,t)|t^{\lambda+\alpha}dt+\int_x^{x+h}\frac{(t-x)^{\lambda+\alpha}}{(x+h-t)^{1+\alpha}}dt+$$

$$+\int_0^x\left|\,(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right||t-x|^{\lambda+\alpha}\Bigg)y^{\nu-1}\int_0^y\frac{(y-s)^{\gamma}}{s^{\nu}}ds+$$

$$+\Bigg(\int_x^{x+h}|B_1(x+h,t)|t^{\lambda+\alpha}dt+\int_0^x|D_1(x,h,t)|t^{\lambda+\alpha}dt\Bigg)\int_0^y\frac{s^{\gamma+\beta}ds}{(y-s)^{1+\beta}}\Bigg\}.$$

Hence, by estimates for F_1 and F_2 from Theorem 1, we have

$$|G_2(x+h,y)-G_2(x,y)|\le C_1h^{\lambda}.$$

The estimate

$$|G_2(x,y+\eta)-G_2(x,y)|\le C_2\eta^{\gamma}$$

is symmetrical obtained.

For the mixed difference $\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)$ with $h,\eta>$ and $x,x+h\in[0,b]$, $y,y+\eta\in[0,d]$ the appropriate representation leading to the separate evaluation in each variable without losses in another variable is as follows:

$$\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)=\int_x^{x+h}\int_y^{y+\eta}B_1(x+h,t)B_2(y+\eta,s)\psi(t,s)dtds+$$

$$+\int_0^x\int_0^y D_1(x,h,t)D_2(y,\eta,s)\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)D_2(y,\eta,s)\psi(t,s)dtds+$$

$$+\int_0^x\int_y^{y+\eta}D_1(x,h,t)B_2(y+\eta,s)\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_0^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-s)^{1+\beta}}\psi(t,s)dtds+$$

$$+\int_0^x\int_y^{y+\eta}\frac{D_1(x,h,t)}{(y+\eta-s)^{1+\beta}}\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_0^y\frac{D_2(y,\eta,s)}{(x+h-t)^{1+\alpha}}\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_0^y B_1(x+h,t)\left[(y+\eta-s)^{-1-\beta}-(y-s)^{-1-\beta}\right]\psi(t,s)dtds+$$

$$+\int_0^x\int_0^y D_1(x,h,t)\left[(y+\eta-s)^{-1-\beta}-(y-s)^{-1-\beta}\right]\psi(t,s)dtds+$$

$$+\int_0^x\int_y^{y+\eta}B_2(y+\eta,s)\left[(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right]\psi(t,s)dtds+$$

$$+\int_0^x\int_0^y D_2(y,\eta,s)\left[(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right]\psi(t,s)dtds+$$

$$+\int_x^{x+h}\int_y^{y+\eta}\frac{B_2(y+\eta,s)}{(x+h-t)^{1+\alpha}}\psi(t,s)dtds.$$

We omit the details of evaluation of each term in the above representation, it is standard via Lemma 2 and yields

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x,y)\right|\le C_3h^{\lambda}\eta^{\gamma}.$$

This completes the proof.

Theorem 4. Let $\rho(x,y)$ have the form $\rho(x,y)=x^{\mu}y^{\nu}$ with $\mu<1+\lambda$, $\nu<1+\gamma$ and let $0<\alpha,\beta<1,\ 0<\lambda,\gamma<1,\ \lambda+\alpha<1,\ \gamma+\beta<1$. Then the operator $J_{0+,0+}^{\alpha,\beta}$ establishes an isomorphism between the spaces $\widetilde{H}_0^{\lambda,\gamma}(\rho)$ and $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$.

Proof. We should consider, as usual the following three parts of the proof:

1) Action of the mixed fractional integral operator from the space $\widetilde{H}_0^{\lambda,\gamma}(\rho)$ to the space $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$;

2) Action of the mixed fractional differentiation operator from the space $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$ to the space $\widetilde{H}_0^{\lambda,\gamma}(\rho)$;

3) The possibility to represent any function $f(x,y)\in\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$ as $(J_{0+,0+}^{\alpha,\beta}\varphi)(x,y)$ with the density in $\widetilde{H}_0^{\lambda,\gamma}(\rho)$.Because of (16) the parts 1) -2) are covered by Theorems 2 and 3. The part 3) is treated in the standard way in case $0<\alpha,\beta<1$ by using the possibility of similar representation with the density from $L_{\bar p}(\mathrm{R}^2)$, $\bar p=(p_1,p_2)$. See [16Theorem 24.4.

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