# Numerical Method of the Line for Solving One Dimensional Initial- Boundary Singularly Perturbed Burger Equation

**Kedir Aliyi, Hailu Muleta**

*Abstract: In this Research Method of Line is used to find the approximation solution of one dimensional singularly perturbed Burger equation given with initial and boundary conditions. First, the given solution domain is discretized and the derivative involving the spatial variable $x$ is replaced into the functional values at each grid points by using the central finite difference method. Then, the resulting first-order linear ordinary differential equation is solved by the fifth-order Runge-Kutta method. To validate the applicability of the proposed method, one model example is considered and solved for different values of the perturbation parameter '$\varepsilon$' and mesh sizes in the direction of the temporal variable, t. Numerical results are presented in tables in terms of Maximum point-wise error, $E_{\varepsilon}^{N,\Delta t}$ and rate of convergence, $P_{\varepsilon}^{N,\Delta t}$. The stability of this new class of Numerical method is also investigated by using Von Neumann stability analysis techniques. The numerical results presented in tables and graphs confirm that the approximate solution is in good agreement with the exact solution.*

*Keywords: Burger equation, perturbation parameter, Method of line, Von Neumann stability analysis.*

## I. INTRODUCTION

Numerical analysis is a subject that involves computational methods for studying and solving mathematical problems. It is a branch of mathematics and computer science that creates, analyzes, and implements algorithms for solving mathematical problems numerically [2]. Also, it's widely used by scientists and engineers to solve some problems. Such problems may be formulated in terms of an algebraic equation, transcendental equations, ordinary differential equations, and partial differential equations [1], [3]. Numerical analysis is also concerned with the theoretical foundation of numerical algorithms for the solution of problems arising in scientific applications [3].

Applications of PDEs can be found in physics, engineering, mathematics, and finance [1], [3], [16]. For instance, include modeling mechanical vibration, heat, sound vibration, elasticity, and fluid dynamics [16].Although PDEs have a wide range of applications to real-world problems in science and engineering, the majority of PDEs do not have analytical solutions. It is, therefore, important to be able to obtain an accurate solution numerically. Many computational methods have been developed and implemented to successfully approximate solutions for mathematical modeling in the application of PDEs. To make use of mathematical models, it is necessary to have solutions to the model equations. Generally, this requires numerical methods because of the complexity and number of equations [4] Scientists in the field of computational mathematics are trying to develop more accurate numerical methods by using computers for further application [16]. One of those numerical methods is a method of line. Burgers' equation, which belongs to the class of Navier–Stokes equation, is a fundamental partial differential equation from the model of fluid mechanics analyses [4, 15]. It was first introduced by Bateman [6]. In 1948, Burgers (1939, 1948) introduced one-dimensional PDEs, to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion; it arises in the theory of shock waves, in turbulence problems, and continuous stochastic processes [12].

The structure of Burgers' equation is roughly similar to that of Navier-Stokes equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. So this equation can be considered as a simplified form of the Navier-Stokes equations [7], [9].

The study of the general properties of the Burgers' equation has attracted the attention of the scientific community due to its applications in various fields such as gas dynamics, heat conduction, elasticity, etc [7].

The study of the solution of Burgers' equation has been carried out for the last half-century and still, it is an active area of research to develop a better numerical scheme to approximate its solution. Due to the wide range of the application of the one-dimensional Burgers equation, several numerical methods have been developed. Even though many numerical methods were applied to solve these types of equations. Accordingly, more efficient and simpler numerical methods are required to solve the Burgers equation. In the literature review, many researchers have used various methods to seek the numerical solutions of 1D Burgers' equation [6]--[10].
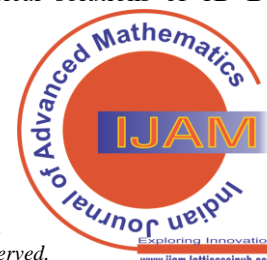
**Kedir Aliyi\*,** Department of Mathematics, College of Natural and Computational Sciences, Ambo University, Ambo, Ethiopia. kediraliyi39@gmail.com

**Hailu Muleta,** Department of mathematics, College of Natural sciences, Jimma University, Jimma, Ethiopia

Similarly, the coupled and 2D Burgers' equations are solved by many researchers with different numerical methods. Abazaria and Borhanifar presented the solution of coupled and 2D Burgers' equations by using the differential transform method (DTM).DTM is a semi-numerical–analytic technique that formalizes the Taylor series differently. The Taylor series method is computationally time-consuming for large orders and high contaminated round-off error and truncation error. Asaithambi [14] presented a Numerical solution to the Burgers' equation by using automatic differentiation. Kutluay, Esen, and Dag in [20] are presented a Numerical solution of the Burgers' equation by the least-squares quadratic B-spline finite element method. Khater [13] proposed the Chebyshev spectral collocation method for solving the coupled Burgers' equations. With pseudo-spectral methods care must be taken with the round-off error issue when higher derivatives or a large several points $N$ is involved. For instance, the utilization of Chebyshev collocation methods incurs a rounding-off error of order ($N2ke$), where $k$ is the order of the PDE and $e$ is the machine zero. This can ruin the computed solution even if $k$ and $N$ are not large. Gowrisankar, S., and Natesan, S. in [15], present the numerical solution of singularly perturbed initial-boundary Burgers' equation by using an efficient robust numerical method. They provide an $e$-uniformly convergent numerical method for the singularly perturbed Burger. They obtain uniform convergence concerning the perturbation parameter $\varepsilon$. Even though the method is capable of approximating Burger's equation, they failed to solve a relatively small perturbation parameter $\varepsilon$. However, still, the accuracy of the method needs attention; because the treatment of the method used to solve the Burger equation is not trivial distribution. Even though the accuracy of the aforementioned methods needs attention, sometimes they require large memory or long computational time besides costing. So the treatments of this method present severe difficulties that have to be addressed to ensure the accuracy of the solution.

To this end, this paper aims to develop a numerical method that is capable of solving singularly perturbed initial-boundary Burger equation for any $\varepsilon$ and approximate the exact solution. The convergence has been shown in the sense of $L_\infty$ norm so that the local behavior of the solution is captured exactly. The stability of the present method is also investigated by using Von Neumann stability analysis techniques.

## II. PRELIMINARIES

### 2.1 Singularly Perturbed Problem

A singular perturbation problem is one for which the perturbed problem is qualitatively different from the unperturbed problem. One typically obtains an asymptotic, but possibly divergent, expansion of the solution, which depends singularly on the parameter $\varepsilon$. Although singular perturbation problems may appear typical, they are the most interesting problems to study because they allow one to understand qualitatively new phenomena.

The solutions of singular perturbation problems involving differential equations often depend on several widely different length of time scales. Such problems can be divided into two broad classes: layer problems, treated using the

method of matched asymptotic expansions (MMAE); and multiple-scale problems, treated by the method of multiple scales (MMS) [15]. Brandt's boundary layer theory for the high Reynolds flow of a viscous fluid over a solid body is an example of a boundary layer problem and the semi-classical limit of quantum mechanics is an example of a multiple-scale problem [15]. An example of the perturbation problem is singularly perturbed Burgers' initial-boundary-value problem. Under suitable continuity and compatibility conditions on the data, the IBVP in EQs (1) has a unique solution, [15].

They used Cole–Hopf transformation which transforms the Burgers' equation to a linear diffusion equation and this diffusion equation can be solved exactly for an arbitrary initial condition with regularity assumption on the initial and boundary conditions. In addition to these, they can assure that boundary layer occurs in the solution when $\varepsilon \to 0$ at the boundary of the domain $x = 1$; the solution varies rapidly, while away from the layer the solution changes slowly, and smoothly. The accuracy of the method is decreased.

Beckett, B, and Mackenzie have presented a numerical solution for one-dimensional convection– and reaction-diffusion problems using equidistribution of the singular component of the solution in [17]. Moreover, space-time parabolic reaction-diffusion and convection-diffusion evolution problems are analyzed by Gowrisankar and Natesan in [18], [19].

### 2.2. The Numerical Method of Lines

The method of lines (MOL) is a convenient procedure for solving time-dependent PDEs, which proceeds in two separate steps: Approximation of the spatial derivatives using finite differences, finite elements, or finite volume methods (or any other techniques), and time integration of the resulting semi-discrete (discrete in space, but continuous in time) ODEs [11].

The method of lines (MOL, NMOL) [5],[22],[23] is a technique for solving partial differential equations (PDEs) in which all but one dimension is discretized [22]. MOL allows standard, general-purpose methods and software, developed for the numerical integration of ODEs and DAEs, to be used [11]. Many integration routines have been developed over the years in many different programming languages and some have been published as open-source resources [24]. The method of lines most often refers to the construction or analysis of numerical methods for partial differential equations that proceeds by first discretizing the spatial derivatives only and leaving the time variable continuous. This leads to a system of ordinary differential equations to which a numerical method for initial value ordinary equations can be applied. The method of lines in this context dates back to at least the early 1960s [11], [25]. Many papers discussing the accuracy and stability of the method of lines for various types of partial differential equations have appeared in [26],[27].

5
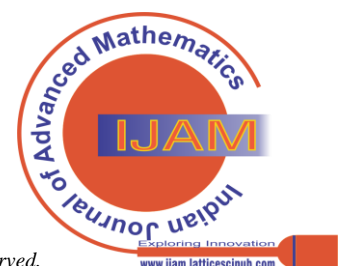
MOL requires that the PDE problem is well-posed as an initial value (Cauchy) the problem in at least one dimension because ODE and DAE integrators are initial value problem (IVP) solvers [11], [19].

Thus it cannot be used directly on purely elliptic partial differential equations, such as Laplace's equation. However, MOL has been used to solve Laplace's equation by using the method of false transients [19]. In this method, a time derivative of the dependent variable is added to Laplace's equation. Finite differences are then used to approximate the spatial derivatives and the resulting system of equations is solved by MOL. It is also possible to solve elliptical problems by a semi-analytical method of lines [29],[30]. In this method, the discretization process results in a set of ODE's that are solved by exploiting properties of the associated exponential matrix. Recently, to overcome the stability issues associated with the method of false transients, a perturbation approach was proposed which was found to be more robust than the standard method of false transients for a wide range of elliptic PDEs [31]

## III. DESCRIPTION OF THE METHOD, RESULTS, AND DISCUSSION

### 3.1 Description of the Method

Consider the following singularly perturbed Burgers' initial-boundary-value problem (IBVP) considered in [15]:

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)(x,t) = \varepsilon\frac{\partial^2 u}{\partial x^2}(x,t),$$

$$(x,t) \in (0,1) \times (0,T] \tag{1}$$

subject to initial and boundary conditions:

$$u(x,0) = f_1(x), \quad 0 \le x \le 1 \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T \tag{3}$$

Here, where $0 < \varepsilon << 1$ is a small perturbation parameter $f_1(x)$, is continuous and differentiable functions. The computational domain $[a,b] \times [0,T]$ is partitioned as: $0 = x_0 < x_1 < ..... < x_j < x_{j+1} < ... < x_M = 1$,

$0 = t_0 < t_1 < ..... < t_n < x_{n+1} < ... < x_N = T$ (4)

$h = x_{j+1} - x_j$ and $\Delta t = t_{j+1} - t_j$ where $h$ and $\Delta t$ are mesh-size of $[0,1]$ and $[0,T]$

### 3.2. Discretizing Partial Derivative involving with Spatial Variable

Recalling that the one-dimensional singularly perturbed Burgers' initial-boundary value problem (IBVP) given in Eq (1), we aim to approximate the partial derivative of $u(x,t)$ involving spatial variable. The given non-linear PDE in Eq(1) is reduced into the system of no linear ODEs by using the method of line. The idea of the method of the line is discretizing partial derivative involving spatial variable by using central finite difference method and the remaining part of variable is discretized.

Now the discretizing of partial derivative involving a spatial variable by using central difference method is:

$$\frac{\partial u}{\partial x} = \frac{u_{j+1,n} - u_{j-1,n}}{2h}, \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{h^2}, \tag{6}$$

where $j = 1,2,3,...,M$ in direction of special variable. Substituting Eqs,(5) and (6) into singularly perturbation Burger's equation given in Eq (1), we obtain the system of the non-linear differential equation of the form:

$$\frac{\partial u}{\partial t}(x_j,t) = \varepsilon\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) - u_j\left(\frac{u_{j+1} - u_{j-1}}{2h}\right) \tag{7}$$

wherein Eq.(7). Hence the given equation is further discretized in space for a first-order and second-order spatial derivative and then obtains a semi discretized scheme corresponding to Burger's equation. In this discretization, we consider redistributing grid points for spatial direction. The distributive of mesh point in the domain, outside and inside of boundary layer region almost equal for both spatial and temporal variables.

### 3.3. Results and Discussion

That method of the line is used to approximate $\frac{\partial u}{\partial x}(x,t)$ and $\frac{\partial^2 u}{\partial x}(x,t)$ by using the central difference method at $N$ grid point in the spatial direction in,$[0,1]$. Then from Eqs (2) and (7) taking into account that the boundary condition in Eq (3) $u_o(t) = 0 = u_M(t)$, the resulting system of nonlinear ODEs with the initial condition given as:

$$\frac{du(x_i,t)}{dt} = \frac{\varepsilon}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) - \frac{u_j}{2h}\left(u_{j+1} - u_{j-1}\right),$$

$$j = 1(1)M \tag{8}$$

subject to the initial condition

$$u(x_i,0) = f_1(x_i), 0 \le x_j \le 1, \tag{9}$$

The system of ODEs in Eq (8) has an $N$ differential equation. Hence by introducing the vector $U$ i.e. $U = [u_1(t), u_2(t), ....., u_M(t)]^T$ in Eq (8), we can rewrite it as matrix form as follow:

$$\frac{dU}{dt} = H(U) \tag{10}$$

$$U_0(x_j) = U_0 \quad , 0 \le x_j \le 1 \quad , \text{j=1(1)M} \quad (11)$$

where $U_0(x_j) = U_0$ initial condition and $H$ is a nonlinear function of $U$ with element is $h_j$ which is given by:

$$h_j(u_1, u_2, \ldots, u_{M-1}, t) = (\alpha - \beta u_j)u_{j+1} - (2\alpha - \beta u_{j-1})u_j + \alpha u_{j-1}$$
$$(12)$$

where $\alpha = \dfrac{\varepsilon}{h^2}$ and $\beta = \dfrac{1}{2h}$ .

As (Gowrrisanker S. and Natesan S.,2019 ) introduced, we consider the discretization of time domain [0, T] the equidistant mashes with uniform time step $\Delta t$ given as:

$$D_n = \{t_n = t_0 + n\Delta t\} \;, \; n = 1(1)N \;,$$

$$\Delta t = \frac{T}{N} \qquad (13)$$

where $N$ is the number of mesh elements in $t$-direction. Then the resulting system of ODEs in Eq (10) can now be solved by using the fifth-order Runge-Kutta method.

### 3.4. Stability Analysis

In this section, the stability of the proposed numerical method is investigated by using Von-Neumann stability analysis. To do these we assumed the non-linear term *uux* of partial differential equation in Eq.(1) as linear by taking $u = \gamma$ where $\gamma$ is constant. Then without losing generality, we obtain the linear system of ODEEs. Assume that $\gamma = \max(u_j)$ in Eq (10). Now we can know to inquire about the eigenvalues of the $N$ system of ODEs (8). To obtain this eigenvalue, as [17]--[20] takes, we assume that a trial solution and substituting it into Eq.(8). However, the trial solution must be taking into account the variation of $u(x,t)$ both $x$ and $t$. This variation of the trial solution is assumed that as in [17], a product of a solution given by:

$$u(x,t) = \varphi(t)\phi(x) \qquad (14)$$

Farther following a method proposed by Von Neumann, we assume that $x$ depending $\phi(x)$ to be of a form:

$$\phi(x) = e^{ik_a x_j} = e^{ijhk_a} \qquad (15)$$

where $i = \sqrt{-1}$ , $k_a = a\pi$ and $a = 1, 2, 3, \ldots, M$ , K. K is a Fourier number or amplification factor. Now substituting Eqs (14) and (15) into Eq (8) we obtain:

$$\frac{d\varphi}{dx}e^{ijhk_a} = \frac{\varepsilon\varphi}{h^2}\left\{e^{ik_a h(j+1)} - 2e^{ik_a hj} + e^{ik_a h(j-1)}\right\} - \frac{\gamma\varphi}{2h}\left\{e^{ik_a h(j+1)} - e^{ik_a h(j-1)}\right\}$$

$$=$$

$$\varphi\left\{\frac{\varepsilon}{h^2}\left[\{\cos(hk_a)+i\sin(hk_a)\}+\{\cos(hk_a)-i\sin(hk_a)\}-2\{\cos(0k_a)-i\sin(0k_a)\}\right]-\right.$$

$$\frac{\varphi\gamma}{2h}\left[\{\cos(hk_a)+i\sin(hk_a)\}+\{\cos(hk_a)-i\sin(hk_a)\}\right]$$

$$= \varphi\left[\frac{\varepsilon}{h^2}\{2\cos(hk_a)-2\} - \frac{i\gamma}{h}\sin(hk_a)\right] \qquad (16)$$

Thus we can write Eq (16) in terms of eigenvalue $\lambda$ such that:

$$\frac{d\varphi}{dt} = \lambda_a\varphi \qquad (17)$$

Therefore from Eqs (16) and (17), we obtain:

$$\lambda_a\varphi = \varphi\left[\frac{\varepsilon}{h^2}\{2\cos(hk_a)-2\} - \frac{i\gamma}{h}\sin(hk_a)\right]$$

$$\lambda_a = \frac{\varepsilon}{h^2}\{2\cos(hk_a)-2\} - \frac{i\gamma}{h}\sin(hk_a) \qquad (18)$$

where $a = 1, 2, 3, \ldots\ldots\ldots\ldots M$ . Hence from Eq (18) we obtain the required egienvalue .All egienvalue has negative real part (i.e Rel($\lambda_a$ )<1). Therefore the obtained system of equation in Eq (11) is stable.

**Theorem 1:-** The obtained system of the equation is stable such that $\lambda$ of the system matrix say matrix 'A' satisfy $\operatorname{Re}al(\lambda) \le 0$ .

**proof:** Assuming that the system matrix is diagonal. Let "P" be invertible matrix. Then, $A = p\lambda p^{-1}$ where $\lambda$ are the eigenvalues of matrix A and

$$\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{N-1} \end{pmatrix} \text{for all } n=1,2,3,\ldots N\text{-}1. \text{ Then}$$

we have:

$$e^{At} = \sum_{n=1}^{\infty}\frac{1}{n!}A^n t^n = \sum_{n=1}^{\infty}\frac{1}{n!}\left(p\lambda p^{-1}\right)^n t^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}p\lambda^n p^{-1}t^n = p\sum_{n=1}^{\infty}\frac{1}{n!}\left(\lambda^n t^n\right)p^{-1} = pe^{\lambda}p^{-1}$$

$$p\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{N-1} \end{pmatrix}p^{-1} \qquad (20)$$

theorem Hence $e^{\lambda t} \to 0$ if and only if the real part of the eigenvalue of "A" is less than zero ( $\operatorname{Re} al(\lambda) \le 0$ ).

This follows that $\left| e^{\lambda t} \right| \to 0$ if and only if $\operatorname{Re} al(\lambda) \to 0$.

Therefore the obtained system of the equation is stable.

### 3.5. Criteria for Investigating the Accuracy of the Method

In this section, we investigate the accuracy of the present method. To show the accuracy of the present method for some values of the perturbation parameter " $\lambda$ ", we report the maximum point-wise absolute *error* $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ The order of convergence and the maximum pointwise absolute error is calculated Gowrisankae and Natesan [15]

$$E_\varepsilon^{N,\Delta t} = \max_{1 \le i \le N} \left| U(x_i, t_M) - u(x_i, t_M) \right|, \qquad (21)$$

$$P_\varepsilon^{N,\Delta t} \log 2 \left( \frac{E_\varepsilon^{N,\Delta t}}{E_\varepsilon^{2N,\Delta t}} \right) \qquad (22)$$

Here, $U(x_i, t_M)$ and $u(x_i, t_M)$ are the exact and approximation solutions of Eqs. (1), (2), and (3), respectively.

### 3.6 Numerical Experiments

To test the validity of the proposed method, we have considered the following model problem.

**Example1**: Consider the one dimensional perturbed Burger equation considered by Gowrisankae and Natesan [15]

$$\left( u_t + u u_{xx} \right)(x,t) = \varepsilon u_{xx}(x,t), \ 0 < x < 1, \ 0 < t \le T,$$

with initial condition

$$u(x_i, 0) = \sin(\pi x_i), \ 0 \le x_i \le 1$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \ 0 \le t \le T,$$

The unique exact solution of the above IBVP Burger's equation is given by :

$$U(x_i, t) = 2\varepsilon\pi \frac{\sum_{p=1}^{\infty} e^{-\varepsilon\pi^2 t} p A_p \sin(p\pi x)}{A_0 + \sum_{p=1}^{\infty} e^{-\varepsilon\pi^2 t} p A_p \cos(p\pi x)}$$

$$A_0 = \int_o^1 \left( e^{\frac{-1}{2\varepsilon} \int_0^x f(y)dy} \right) dx \quad \text{and}$$
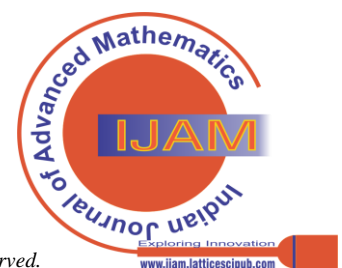
$$A_n = \int_o^1 \left( \left[ e^{\frac{-1}{2\varepsilon} \int_0^x f(y)dy} \right] \cos(p\pi x) \right) dx$$

The numerical results are presented in tables in terms of $E_\varepsilon^{N,\Delta t}$ and $p_\varepsilon^{N,\Delta t}$, measuring the accuracy of the present method for different values of perturbation parameter $\varepsilon$ .

**Table 1. Maximum Pointwise absolute error $E_\varepsilon^{N,\Delta t}$ and rate of convergence $p_\varepsilon^{N,\Delta t}$ example1 on equidistribution mesh.**

Our Method

| $N/\Delta t \to$ | $64 / \frac{1}{20}$ | $128 / \frac{1}{40}$ | $256 / \frac{1}{80}$ | $512 / \frac{1}{160}$ |
|---|---|---|---|---|
| $\varepsilon \downarrow$ | $E_\varepsilon^{N,\Delta t} / P_\varepsilon^{N,\Delta t} \downarrow$ | | | |
| $10^0$ | 4.8019E-05 | 1.8803E-5 | 8.3113 E-06 | 9.00 E-06 |
| | 0.3566 | 1.8803 | 0.0900 | 0.0455 |
| $10^{-2}$ | 5.9723E-02 | 3.0160 E-02 | 1.5018 E-02 | 7.4764E-03 |
| | 0.0375 | 0.0191 | 0.0095 | 0.0048 |

| $\varepsilon$ | 64 / 1/20 | 128 / 1/40 | 256 / 1/80 | 512 / 1/160 |
|---|---|---|---|---|
| $10^{-4}$ | 1.0700E-02 0.0061 | 1.4224 E-02 0.0082 | 1.2081 E-02 0.0070 | 7.2973E-03 0.0042 |
| $10^{-6}$ | 1.2349E-04 -0.0103 | 2.5231E-04. -0.000774 | 4.9521 E-04 0.000178 | 9.0564E-04 0.000051 |

By Gowrisankae,S. and Natesan,S. ,2019 in [15]

| $\varepsilon$ | 64 / 1/20 | 128 / 1/40 | 256 / 1/80 | 512 / 1/160 |
|---|---|---|---|---|
| $10^{0}$ | 7.5816E-03 0.9903 | 3.8165E-03 0.9877 | 1.924E-03 0.9875 | 9.7064E-04 — |
| $10^{-2}$ | 1.1303E-01 0.8700 | 6.1846E-01 1.1644 | 2.7592E-02 0.9714 | 1.4072E-02 — |
| $10^{-4}$ | 2.5946E-01 0.8477 | 1.4418E-01 1.0752 | 6.8426E-02 1.0753 | 3.2474E-02 — |
| $10^{-6}$ | 2.7194E-01 0.7955 | 1.5667E-01 1.0432 | 6.9863E-02 1.0432 | 3:2474E-02 — |

**Table 2. Maximum Pointwise absolute error $E_{\varepsilon}^{N,\Delta t}$ and rate of convergence $p_{\varepsilon}^{N,\Delta t}$ example1 on a uniform mesh**

Our Method

| $N/\Delta t \rightarrow$ $\varepsilon \downarrow$ $\left. E_{\varepsilon}^{N,\Delta t} \middle/ P_{\varepsilon}^{N,\Delta t} \right. \downarrow$ | $64 \big/ \dfrac{1}{20}$ | $128 \big/ \dfrac{1}{40}$ | $256 \big/ \dfrac{1}{80}$ | $512 \big/ \dfrac{1}{160}$ |
|---|---|---|---|---|
| $10^{0}$ | 4.8019E-05 0.3566 | 1.8803E-5 1.8803 | 8.3113 E-06 0.0900 | 9.00 E-06 0.0455 |
| $10^{-2}$ | 5.9723E-02 0.0375 | 3.0160 E-02 *0.0191* | 1.5018 E-02 0.0095 | 7.4764E-03 0.0048 |
| $10^{-4}$ | 1.0700E-02 0.0061 | 1.4224 E-02 0.0082 | 1.2081 E-02 0.0070 | 7.2973 E-03 0.0042 |

| $\varepsilon$ | | | | |
|---|---|---|---|---|
| $10^{-6}$ | 1.2349E-04 | 2.5231E-04. | 4.9521 E-04 | 9.0564 E-04 |
| | -0.0103 | -0.000774 | 0.000178 | 0.000051 |

By Gowrisankae,S. and Natesan,S. ,2019 in [15]

| $\varepsilon$ | | | | |
|---|---|---|---|---|
| $10^{0}$ | 7.5816E-03 | 3.7677 E-03 | 1.8924 E-03 | 9.4836 E-04 |
| | 0.9903 | 0.9935 | 0.9966 | — |
| $10^{-2}$ | 1.0753 E-01 | 7.6588 E-02 | 0.5937 E-02 | 0.7621 E-02 |
| | 0.4895 | 0.5937 | 0.7621 | — |
| $10^{-4}$ | 9.7343 E-02 | 8.3641 E-02 | 7.2542 E-02 | 6.6633$E$-02 |
| | 0.2188 | 0.2054 | 0.1226 | — |
| $10^{-6}$ | 9.5552 E-02 | 7.9607 E-02 | 6.4438 E-02 | 0.3174 E-02 |
| | 0.2634 | 0.305 | 0.3174 | — |

**Table 3. Maximum Pointwise absolute error $E_{\varepsilon}^{N,\Delta t}$ and rate of convergence $p_{\varepsilon}^{N,\Delta t}$ example1 on Shishkin mesh**

Our Method

| $N/\Delta t \rightarrow$ <br> $\varepsilon \downarrow$ | $64 / \dfrac{1}{20}$ <br> $E_{\varepsilon}^{N,\Delta t} / P_{\varepsilon}^{N,\Delta t} \downarrow$ | $128 / \dfrac{1}{40}$ | $256 / \dfrac{1}{80}$ | $512 / \dfrac{1}{160}$ |
|---|---|---|---|---|
| $10^{0}$ | 1.2041E-04 | 4.7393 E-5 | 2.0979 E-05 | 2.0078 E-05 |
| | 0.3555 | 0.1778 | 0.088 | 0.0456 |
| $10^{-2}$ | 1.1647E-01 | 6.8416 E-02 | 3.5531 E-02 | 7.4764E-03 |
| | 0.0307 | 0.0180 | 0.0094 | 0.0048 |
| $10^{-4}$ | 5.1305E-03 | 9.35354 E-03 | 1.3651 E-03 | 1.3621 E-03 |
| | 0.000999 | | 0.0032 | |
| | | 0.0022 | | 0.00162 |
| $10^{-6}$ | 4.1915E-05 | 1.0490 E-04. | 2.1141 E-04 | 2.1180 E-04 |
| | -0.0256 | -0.0024 | -0.000251 | -0.000021 |

By Gowrisankae,S. and Natesan,S. ,2019 in [15]

| $10^0$ | 7.4679E-03 | 3.7677E-03 | 1.8924 E-03 | 9.4836E-04 |
|---|---|---|---|---|
| | 0.9870 | 0.9935 | 0.9966 | — |
| $10^{-2}$ | 9.5492E-02 | 5.580 4E-0 | 3.2269E-02 | 1.7885 E-02 |
| | 0.7750 | 0.7502 | 0.8513 | |
| | | | | — |
| $10^{-4}$ | 6.4274E-01 | 5.1317E-01 | 3.4536 E-01 | 2.0605 E-01 |
| | 0.3248 | 0.5713 | 0.7450 | — |
| $10^{-6}$ | 8.0790E-01 | 8.8692 E-01 | 8.2825 E-01 | 6.9417E-01 |
| | -0.1346 | 0.9873 | 0.2547 | — |
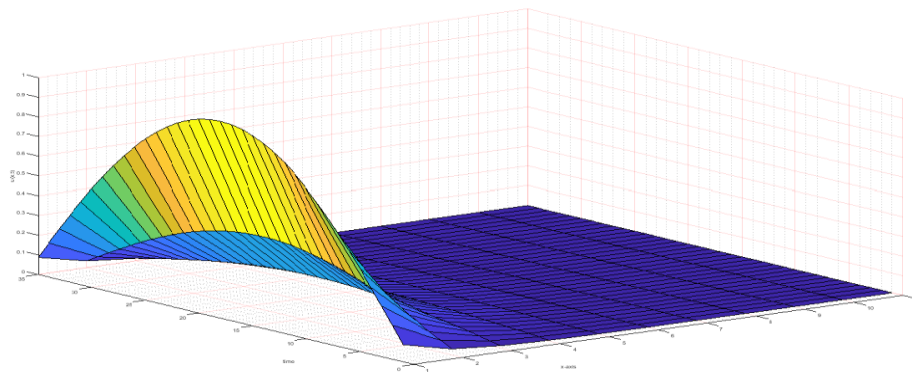


**Figure 1: Solution profile of Example 1on uniform mesh with ε=2⁻², M = 36 & Δt=0.1**
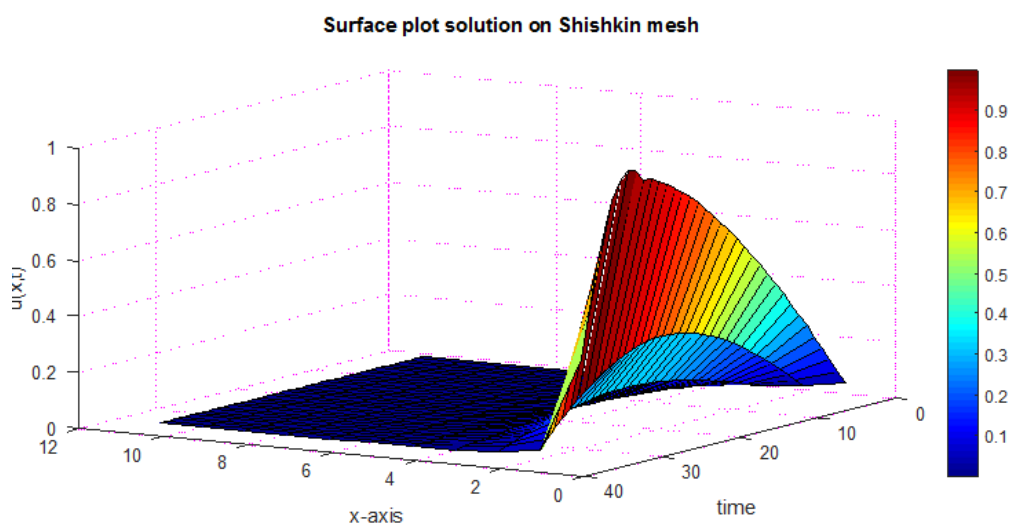


**Figure 2:Solution profile of Example 1on shrinking mesh with ε=2⁻⁶ ,M = 36 & Δt=0.1**

11
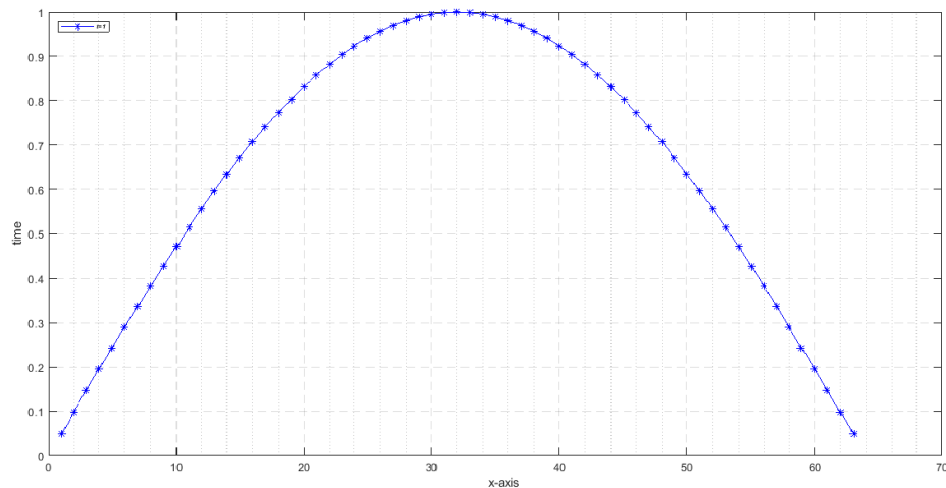
**Figure 3: Graphs for the numerical solution of Example 1 on uniform mesh with ε=10-$^{-2}$, M = 64 & Δt=0.1 to show the point where the maximum pointwise error are obtained**
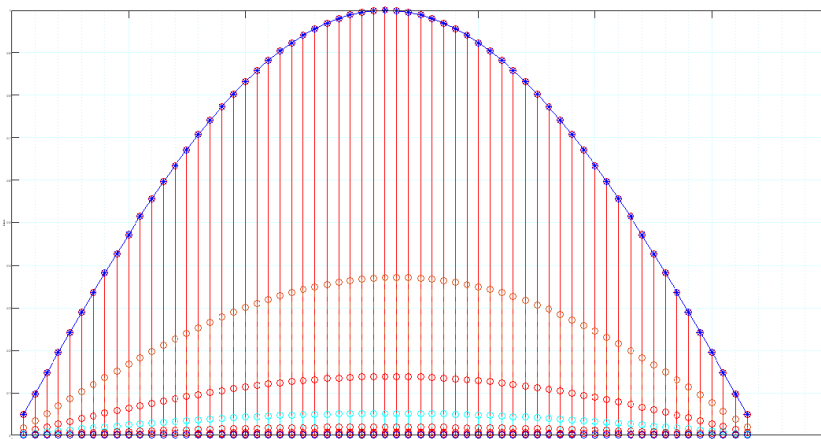


**Figure 4: Graphs of the numerical solution and Grid distribution for Example 1 uniform mesh with ε=10$^{-2}$, M = 64 & Δt=0.1 to show the point where the solution is obtained**
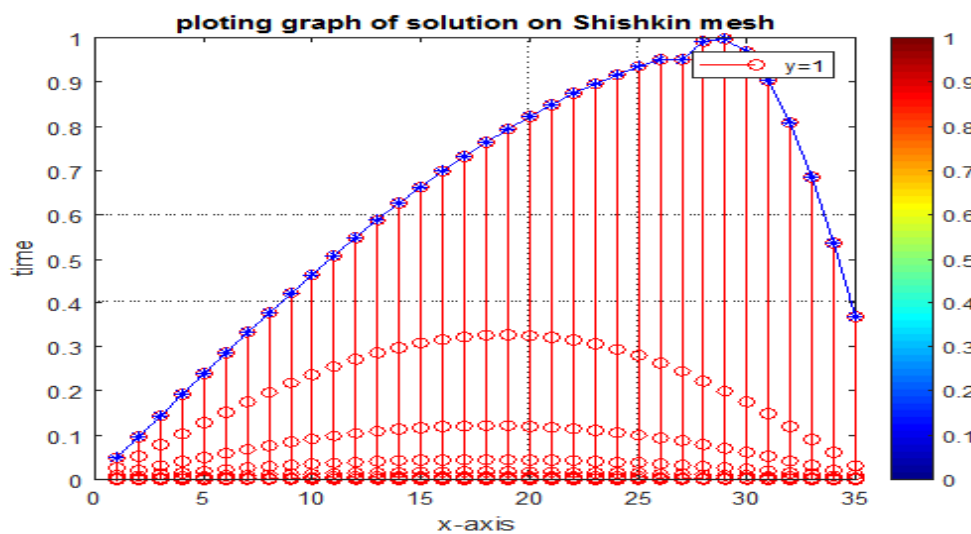


**Figure 5:Graphs of numerical solution and Grid distribution for Example 1 shrinking mesh with ε=10$^{-02}$, M = 36 & Δt=0.1 to show the point where the solution is obtained**

## IV. . DISCUSSION

As depicted in Tables 1, 2, and 3 the present method can generate a convergent numerical solution for $\varepsilon$ and for different mesh refinement at which the method presented by Gowrisankae and Natesan, 2019 in [15] fails to produce the convergent solution. To compare the results obtained by the present method with the already existing methods, we produced maximum point-wise errors and the corresponding order of convergence applying on the uniform mesh in the spatial direction in Table 2, and Shishkin mesh results are given in Table 3. From the results given in both Tables, one can observe that when $\varepsilon = 10^{-04}$ both uniform and Shishkin meshes give high convergence than the already existing method. As the value of the perturbation parameter decrease, the accuracy of the numerical result increase. Further, the numerical results present in Tables 2 and 3 show that the accuracy of the method is enhanced for different values of time-step and step length h for both uniform and Shishkin meshes in the spatial direction. The convergence of the present method is also depending on the perturbation parameter. The parameter-uniform convergence of the scheme is validated with numerical results. To Shaw, the physical behavior of the given problem, the surfaces, and plot graphs of approximating solution are given in Figures 1, 3, and 4, for $\varepsilon = 10^{-02}$, M = 64 & $\Delta t$=0.1. Finger 4 shows that the sequence of the line that the method is used to give the better solution through line segment for the uniform mesh. It means that the distribution of the solution on the grid pint along the line segment. For parameter $\varepsilon$ Comparison among Table 1-Table 3 shows that a more accurate result is generated by the present method. Again the presented in Figures 1, 3, and 4 shows that the approximate solution obtained by the present method for uniform mesh is in good agreement with the exact solution. The simulations presented in Figures 2 and 5 shows that the approximate solution obtained by the present method for shrinking mesh is also in good agreement with the exact solution.

## V. CONCLUSION

In this paper, the 'numerical method of the line' is used to solve one-dimensional singularly perturbed Burger's equation. First, we discretizing the derivative involving the special variable by using the central difference method to obtain the system of ODE. Then we solve the resulting initial value problem by using the Runge-Kutta method and the stability of the method is also established.

To validate the applicability of the method, one model example is considered and solved by varying the value of perturbation parameter ε, time-step k, and step-length h. As can be seen from the numerical results presented in tables and graphs, the present method is superior to the method developed by Gowrisankae and Natesan, 2019 in [15]and approximates the exact solution very well. In a nutshell, the present method is conceptually simple, easy to use, and adaptable for computer implementation for solving one-dimensional singularly perturbed Burger's equation.

## REFERENCES

1. S.S.Sastry.Introductory method of numerical analysis, Fourth-edition, Asoke. k.Ghash, prentice Hall of India, 2006.
2. S. S. Ray.Numerical Analysis with algorithm and programming, 1st edition, Chapman and Hall/CRC- Press, USA, 2016.
3. F. B. Hildebrand .Introduction to numerical analysis, Second –edition, Dover Publications- Inc.,Canada, 1987.
4. William E. Schiesser(2009)A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab,United States of America , Cambridge University Press, New York. [CrossRef]
5. Samir Hamdi, William E. Schiessery and Graham W. Griffiths .Method of Lines,Ecole Polytechnique, France; yLehigh University, USA; zCity , University,UK, 2009.
6. R.C. Mittal, P. Singhal. Numerical solution of Burgers' equation, Comm. Numer. Methods Engrg. , 1993, Vol. 9, pp. 397-406. [CrossRef]
7. R. C.Mittal, R. Jiwari. A differential quadrature method for numerical solutions of Burgers'-type equations. International Journal of Numerical Methods for Heat and Fluid Flow, 2012, 22(7), pp: 880-895. [CrossRef]
8. [8] T. Ozis, E.N. Aksan, and A. Ozdes, ."A finite element approach for solution of Burgers' equation", Appl. Math. Comput, 2003, Vol. 139, pp. 417-28 [CrossRef]
9. M.K. Kadalbajoo, and A. Awasthi. "A numerical method based on CrankNicolson sc heme for Burgers' equation", Appl. Math. Comput., , 2006, Vol. 182, pp.1430-42[CrossRef]
10. S. Kutluay, A.R. Bahadir, and Ozdes, A. "Numerical solution of one dimensional Burgers' equation: explicit and exact-explicit finite difference methods", J. Comput. Appl. Math., 1999, Vol. 103, pp. 251-61 [CrossRef]
11. B .Michael. The Numerical Method of Lines for Partial Differential Equations Cutlip, University of Connecticut and Mordechai Sachem, Ben-Gurion University of the Negev.
12. J. M.Burgers. A mathematical model illustrating the theory of turbulence. In Advances in applied mechanics, Elsevier. 1948, Vol. 1, pp. 171-199. [CrossRef]
13. Khater, A. H., Temsah, R. S., and Callebaut, D. K. (2008). Numerical solutions for some coupled nonlinear evolution equations by using spectral collocation method. Mathematical and Computer Modelling, 48(7-8), 1237-1253. [CrossRef]
14. A .Asaithambi,. "Numerical solution of the Burgers' equation by automatic differentiation. Applied Mathematics and Computation, 2010, 216(9), 2700-2708. [CrossRef]
15. S., Gowrisankar, S. Natesan. "An efficient robust numerical method for singularly perturbed Burgers' equation" . Applied Mathematics and Computation , 2019, 346, 385-39[CrossRef]
16. D. Waston. "Radial Basis Function Differential Quadrature Method for the numerical solution of partial differential equations". The Aquila Digital Community, Dissertations , 2017.
17. Beckett,B,and Mackenzie .Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value prob lem, Appl. Numer. Math.,2000, 35 (2) 87–109 [CrossRef]
18. S. Gowrisankar, and S. Natesan . "The parameter uniform numerical method for singularly perturbed parabolic reaction–diffusion problems on equidistributed grids", Appl. Math. Lett. ,2013, 26 (11) 1053–1060. [CrossRef]
19. S.Gowrisankar, and S. Natesan .Uniformly convergent numerical method for singularly perturbed parabolic initial-boundary-value problems with equidis tributed grids, Int. J. Comput. Math. 2014b, 91 (3) , 553–577[CrossRef]
20. S. Kutluay, A. Esen, I. Dag . "Numerical Solution of the Burgers' equation by the least- squares quadratic B-spline finite element method" , J. Comput. Appl. Math. ,2004, 167 (2004) 21–33[CrossRef]

21. Alain Vande Wouwer . Philippe Saucez Carlos Vilas "Simulation of ODE/PDE Models with MATLAB, OCTAVE and SCILAB,Spri nger International Publishing Switzerland, 2014[CrossRef]
22. W. E. Schiesser, G. W. Griffiths. "A Compendium of partial differential equation models: method of lines analysis with Mat lab" . Cambridge University Press, 2009. [CrossRef]
23. S. Hamdi, W. E Schiesser, G. W. Griffiths, Method of lines. Scholar podia, 2007, 2(7), 2859 [CrossRef]
24. H. J. Lee, W. E. Schiesser,"Ordinary and partial differential equation routines in C, C++, Fortran, Java, Maple, and Matlab". Chapman and Hall/CRC. 2003.
25. E. N. Sarmin, L. A. Chudov ."On the stability of the numerical integration of systems of ordinary differential equations arising in the use of the straight line method", USSR Computational Mathematics and Mathematical Physics ,1963, 3(6), 1537-1543. [CrossRef]
26. A. Zafarullah. "Application of the method of lines to parabolic partial differ entail equations with error estimates. Journal of the ACM (JACM), 1970, 17(2), 294-302. [CrossRef]
27. J. G .Verwer, J. M. Sanz-Serna. "Convergence of method of linen approximations to partial differential equations. Computing, 1984,33(3-4), 297-313. [CrossRef]
28. Schiesser, W. E. (2014). Computational mathematics in engineering and applied science: ODEs, DAEs, and PDEs. CRC press. [CrossRef]
29. V. R. Subramanian, R. E. White . "Semianalytical method of lines for solving elliptic partial differential equations. Chemical engineering science, 2004, 59(4), 781-788[CrossRef]
30. R. E .White, V. R. Subramanian. "Method of Lines for Elliptic Partial Differential Equations. In Computational Methods in Chemical Engineering with Maple, Springer, Berlin, Heidelberg.2010, pp. 507-585. [CrossRef]
31. P. W. Northrop, P. A .Ramachandran, W. E. Schiesser,R Subramanian. "A robust false transient method of lines for elliptic partial differential equations". Chemical Engineering Science,2013, 90, 32-39[CrossRef]

## AUTHORS PROFILE

**Kedir Aliyi Koroche**, is Graduate B.sc in Mathematics from Ambo University in 2016 G.C. and M. Sc. in Mathematics (specialized in Numerical Analysis) from Jimma University in 2019 G.C. Now he works at Ambo University as Lecturer and Researcher. He conducts research on Numerical Analysis. His topics of research in numerical Analysis include: Numerical solutions of ODEs, PDEs using single step and Multistep methods, Finite difference, Differential Quadrature methods.

**Hailu Muleta chemeda,** is lecturer (Assistant Professor of Mathematics (Numerical Analysis)) at Jimma University and head of Students' affairs and registrar of College of Natural Sciences. Besides teaching and office duties, he conducts research on Mathematics Education and Numerical Analysis. His topics of research in numerical Analysis include: Numerical Linear Algebra, Numerical solutions of ODEs, PDEs using Finite difference, Radial Basis-Quadrature and Differential Quadrature methods.

14