Modified Moments and Maximum Likelihood Estimators for Parameters of Erlang Truncated Exponential Distribution

Kannadasan Karuppaiah, Vinoth Raman

Abstract: ***This study derives the parameter estimation in truncated form of a continuous distribution which is comparable to Erlang truncated exponential distribution. The shape and scale parameter will predict the whole distributions properties. Approximation will be useful in making the mathematical calculation an easy understand for non-mathematician or statistician. An explicit mathematical derivation is seen for some properties of, Method of Moments, Skewness, Kurtosis, Mean and Variance, Maximum Likelihood Function and Reliability Analysis. We compared ratio and regression estimators empirically based on bias and coefficient of variation.***

Keywords: Erlang-Truncated Exponential Distribution, Maximum Likelihood Function, Moments, Moment Generating Function, Reliability Analysis.

I. INTRODUCTION

A Random Variable (r.v) X following Erlang Truncated Exponential Distribution (ETED), if its density function (p.d.f) is given by

$$f(x)_{ETED} = \theta\left(1-e^{-\lambda}\right)e^{\theta x\left(1-e^{-\lambda}\right)} \qquad x \geq 0, \theta, \lambda > 0 \qquad \dots(1)$$

The corresponding the Probability distribution function is

$$F(x)_{ETED} = 1 - e^{-\theta x\left(1-e^{-\lambda}\right)} \qquad \dots(2)$$

And corresponding Survival Function (S.F) is

$$\bar{H}(x)_{ETED} = e^{-\theta x\left(1-e^{-\lambda}\right)} \qquad \dots(3)$$

ETED is commonly seen in the area of queuing and stochastic processes. This family of distribution derived from the exponential distribution. More details study can be seen in (Ei-Alosey, 2017) who initially introduced this (ETED), (Ahsanullah, M 1991 and Ahsanullah, M 1992) has estimated the distributional properties of the parameter archives through Lomax Distribution.

(Balakrishnan, 1994) derived the repetition relations for moments estimation of record values through Gumbel distribution. (Ahsanullah.M, 1995) derived some moments of the distribution properties in each records.In this Study we have derived and estimated the MLE on Erlang Truncated Exponential model and discussed some of its statistical properties. This study is structured as follows: Moment Generating function is given in section 2.1, Method of Moments, Skewness, Kurtosis, Mean and Variance has been given in section 2.2, Maximum Likelihood Function in section 2.3 and Reliability analysis is specified in section 2.4.The inference in section 3.

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II. MATHEMATICAL FORMULATION

2.1 Moment Generating Function (M.G.F)

This sub-section derived the (m.g.f) of ETED

$$E(e^{tx}) = \int_0^\infty e^{tx}\ f(x)dx \qquad \ldots(4)$$

$$= \int_0^\infty e^{tx}\ \theta(1-e^{-\lambda})e^{\theta x\,(1-e^{-\lambda})dx}$$

$$= \theta(1-e^{-\lambda})\int_0^\infty e^{tx-\theta x\,(1-e^{-\lambda})}\ dx$$

$$= \frac{\theta(1-e^{-\lambda})}{\theta(1-e^{-\lambda})-t}$$

$$= \left[1-\frac{t}{\theta(1-e^{-\lambda})}\right]^{-1} \qquad \ldots(5)$$

2.2 Method of Moment Estimation

The r.v derived in moments $\bar{X} = X_1, X_2, X_3, \ldots X_n$ as a random sample on X having p.d.f indicated in (5). The likelihood corresponding to the sample is

$$E^r = 1 + \frac{t}{\theta(1-e^{-\lambda})} + \frac{t^2}{\theta^2(1-e^{-\lambda})^2} + \cdots \qquad \ldots(6)$$

$$\mu_1' = \frac{1}{\theta(1-e^{-\lambda})} \qquad \ldots(7)$$

$$\mu_2' = \frac{2}{\theta^2(1-e^{-\lambda})^2} \qquad \ldots(8)$$

$$\mu_3' = \frac{6}{\theta^3(1-e^{-\lambda})^3} \qquad \ldots(9)$$

$$\mu_4' = \frac{24}{\theta^4(1-e^{-\lambda})^4} \qquad \ldots(10)$$

Based on the equation (6), coefficient of Skewness, Kurtosis and Mean, Variance; ETED been attained through the following relation

$$\mathrm{SKE_{ETED}} = \frac{4/\theta^6(1-e^{-\lambda})^6}{1/\theta^6(1-e^{-\lambda})^6} \qquad \ldots(11)$$

$$\mathrm{KUR_{ETED}} = \frac{9/\theta^4(1-e^{-\lambda})^4}{1/\theta^4(1-e^{-\lambda})^4} \qquad \ldots(12)$$

$$\mathrm{E(X)_{ETED}} = \frac{1}{\theta(1-e^{-\lambda})} \quad \dots(13)$$

$$\mathrm{V(X)_{ETED}} = \frac{1}{\theta^2(1-e^{-\lambda})^2} \quad \dots(14)$$

2.3 Maximum Likelihood Function

The R.V derived through likelihood; $\bar{X} = X_1, X_2, X_3, \dots X_n$ as a random sample on X having p.d.f indicated in (1). The likelihood corresponding to the sample is

$$L = \prod_{i=1}^{n} \theta\left(1-e^{-\lambda}\right) e^{\theta x\,(1-e^{-\lambda})} \quad \dots(15)$$

$$= \theta^n\left(1-e^{-\lambda}\right)^n e^{-\theta\,(1-e^{-\lambda})\sum_{i=1}^{n} x}$$

The log likelihood becomes

$$\log L = \log\theta^n + \log\left(1-e^{-\lambda}\right)^n - n\theta\mu\,\left(1-e^{-\lambda}\right)$$

Maximum Likelihood Estimator (MLE) is derived as

$$\frac{\partial \log L(\theta/x)}{\partial\theta} = 0$$

ML Estimator θ is solution of

$$\frac{\partial \log \mathrm{L}}{\partial\theta} = \frac{n}{\theta} - n\mu\,\left(1-e^{-\lambda}\right) \quad \dots(16)$$

ML Estimator λ is solution of

$$\frac{\partial \log \mathrm{L}}{\partial\lambda} = \frac{n\lambda e^{-\lambda}}{(1-e^{-\lambda})} - n\theta\mu\lambda\, e^{-\lambda} \quad \dots(17)$$

ML Estimator μ is solution of

$$\frac{\partial \log \mathrm{L}}{\partial\mu} = n\theta\,\left(1-e^{-\lambda}\right) \quad \dots(18)$$

MLE, of θ, λ and μ is attained through the nonlinear classification of equations. The location of these parameters to zero and solving simultaneously yields the MLE of three parameters. To obtain, elements of information matrix, we will get the second order derivative of logarithms of the likelihood functions as follow:

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}$$

$$\frac{\partial^2 \log L}{\partial\lambda^2} = \frac{\left(1-e^{-\lambda}\right)\left(e^{-\lambda}+\lambda e^{-\lambda}(-\lambda)\right) - \lambda^2 e^{-\lambda}\left(e^{-\lambda}\right)}{(1-e^{-\lambda})^2}$$

$$= \frac{e^{-\lambda} - \lambda^2 e^{-\lambda} - e^{-2\lambda}}{(1-e^{-\lambda})^2}$$

$$= \frac{e^{-\lambda}\left(1-\lambda^2-e^{-\lambda}\right)}{(1-e^{-\lambda})^2}$$

$$\frac{\partial^2 \log L}{\partial\mu^2} = 0, \quad \frac{\partial^2 \log L}{\partial\theta\,\partial\mu} = -n\,\left(1-e^{-\lambda}\right), \quad \frac{\partial^2 \log L}{\partial\theta\,\partial\lambda} = -n\mu\lambda e^{-\lambda}, \quad \frac{\partial^2 \log L}{\partial\lambda\,\partial\mu} = -n\theta\lambda e^{-\lambda}$$

$$\frac{\partial^2 \log L}{\partial\lambda\,\partial\theta} = -n\mu\lambda e^{-\lambda}, \quad \frac{\partial^2 \log L}{\partial\mu\,\partial\theta} = n\left(1-e^{-\lambda}\right), \quad \frac{\partial^2 \log L}{\partial\mu\,\partial\lambda} = n\theta\lambda e^{-\lambda}$$

For interval estimation of (θ, λ, μ) the observed information matrix is obtained since, its probability involves mathematical combination. The 3×3, observed information matrix $J(\theta)$ is

$$J(\theta) = -\begin{pmatrix} I_{\theta\theta} & I_{\theta\lambda} & I_{\theta\mu} \\ I_{\lambda\theta} & I_{\lambda\lambda} & I_{\lambda\mu} \\ I_{\mu\theta} & I_{\mu\lambda} & I_{\mu\mu} \end{pmatrix}$$

Whose elements are given above, under regularity conditions, the asymptotic distribution of

$$\sqrt{n}\,(\hat{\theta} - \theta) \text{ is } N_3\,(0, I\,\theta^{-1})$$

Where, $I(\theta)$ is the estimated information matrix. This asymptotic distribution performance is effective if $I(\theta)$ is substituted by $J(\theta\,)$, i.e., the observed information matrix assessed at $\hat{\theta}$.

2.4 Reliability Analysis

The S.F, also identified as reliability function in engineering, is the representative of an explanatory variable that plots a set of measures, generally related with failure of some structure onto time. We assume, the probability that the structure will survive beyond an indicated time.

Reliability function $R(t)$, is the chance of an item not failing prior to a one-time t, which is defined by; $R(t) = 1 - F(t)$. The $R(t)$of an ETED is assumed as

$$R(t) = 1 - e^{-\theta x(1-e^{-\lambda})}$$

Other parameterconcern of an r.v is the hazard rate function assumed by

$$h(t) = \frac{f(t)}{1 - F(t)}$$

This is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The hazard rate function for an ETED, r.v is given by

$$h(t) = \frac{\theta(1 - e^{-\lambda})e^{\theta x\,(1-e^{-\lambda})}}{1 - e^{-\theta x(1-e^{-\lambda})}}$$

III. CONCLUSION

We derived some expression and recurrence for moments and M.G.F to observe, values of ETED. Further, the estimation of parameters is derived by method of ML and obtained the information matrix.

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